ISSN 2709-4170 (Print)
ISSN 2791-0709 (online)
https://doi.org/10.52587/tmcs030101
Transactions in Mathematical and Computational Sciences
Vol. 03(1)(2024) pp. 01–15

CHARACTERIZATION OF COMPLETE P-ARY TREE WITH DEGREE-BASED TOPOLOGICAL DESCRIPTORS

NOHA MOHAMMAD SEYAM

Department of Mathematical Sciences, College of Applied Sciences, Umm Al-Qura University, Saudi Arabia

Abstract. Topological index is an important numerical magnitude that can reflect the whole structure of a graph. Degree-based indices are mathematical descriptors worked in chemical graph theory to compute the connectivity and underlying descriptions of molecules. These indices, such as the Wiener index, Randić index, and Zagreb indices, are obtained from the quantities of vertices in molecular graphs. The Wiener index correspond to the calculation of gaps between all pairs of nodes in a graph, presenting information about molecular dimension and separating. Zagreb indices, including the first and second Zagreb indices, summarize the measure allocation and node connectivity within a molecular graph. Degree-based indices portray a critical position in molecular modeling, QSAR investigations, and medication layout, requiring perceptions into molecular regional anatomy and forecasting various physicochemical assets. In this article, we compute topological descriptors for complete p-ary trees. Interesting comparison of these indices are is also shown in tabular and graphical format. Moreover, expressions for multiple Zagreb indices and polynomials for these important classes are found.
Keywords: $p-$ary tree topological indices, multiple Zagreb indices and Zagreb polynomials

1. Introduction

Chemical graph theory works as a convincing context for understanding the structure, properties, and activities of molecules. By depicting chemical structures as graphs and applying graph theory based simplifications, researchers obtain valuable intuitions into chemical connectivity, molecular descriptors, aromaticity, isomerism, and equilibrium. The interdisciplinary environment of chemical graph theory forwards alliances between mathematicians, chemists, and computer scientists, steering advance and discovery in fields extending from drug discovery to materials science. Being a computational formulae and methodologies continue to advance, molecular graph theory will continue to help

*Correspondence: nmseyam@uqu.edu.sa
Received: 13.05.2024 Revised: 22.05.2024 Accepted: 28.05.2024 Published: 10.07.2024

out in molecular research, making scientists to discover the broad boundaries of chemicals theory and address complex tasks in chemistry and also its applications in other fields.

Providing worthy comprehensions into physicochemical and structural possessions, topological numbers are numerical resulting notions developed from the chemical graphs of a molecular combination. These numbers play an important task in QSPAR studies which are called as Quantitative Structure-Activity Relationship studies [1], drug theory and in molecular developing. In recent past years, many topological numbers have been improved to portray different aspects of chemical topology. Such as cycles, symmetry and branching as well. Introduced by Harold Wiener in 1947 [2], among all these numerics, the Wiener index, see for details [3, 4].

After the Wiener index, the Randić number, developed by Milan Randić in 1975 [5], and the it is defined by:

$$R_\alpha = \sum_{\lambda_i \lambda_{\in} E(G)} (d_{\lambda_i} d_{\lambda_j})^\alpha. \tag{1}$$

Where α is a real number. The Balaban index, introduced by Dimitrije Balaban in 1982 [6], determines the molecular difficulty and aromaticity based on the distances between pairs of vertices. Furthermore, the Zagreb indices, defined by Ante Graovac and Nenad Trinajstic in 1972, require information about the degree allocation and vertex connectivity in a molecular graph. These topological indices serve as powerful equipment for calculating physicochemical properties, biological behaviour, and molecular behavior, thereby showing drug discovery attempts and accelerating rational drug design. Besides, the change and claim of topological indices have underwrote to the progression of computational chemistry and cheminformatics, allowing researchers to analyze and understand complex molecular structures with better efficiency and accuracy. Some topological indices are given in Table 1.

Topological indices, initially settled for molecular diagrams, have found applications in separate fields alternating from chemistry to computer science. Although utmost explore on topological indices has persistent on molecular diagrams, there is also attention in applying these indices to tree configurations [7]. Trees, as a identifiable type of graph with classified group, share some connections with molecular diagrams, granting certain topological indices to be tailored and employed to tree formations. Topological indices overture valuable equipment for studying classified organization within trees, requiring quantitative procedures of structural appearance such as branching patterns, levels of ladder, and overall connectivity [5, 8]. Furthermore, these indices enable resemblance and organization of tree assemblies, assisting tasks like gathering similar trees and finding common operational topics [9]. Besides, they aid in investigating the advance and growth of trees over time, suggesting awareness into evolutionary developments and departure patterns.

In networking, topological indices boost the program of tree-based communication networks by studying connectivity and good organization [10]. Likewise, in machine learning, these indices evaluate the underlying things of decision trees, grassing outcomes about model collection and performing assessment. Although modifications are obligatory for direct relevance to trees, topological indices provide valuable comprehensions into the association, development, and functionality of tree-based arrangements through several areas [11].

TABLE 1. Some Topological indices (TIs)

Sr. No.	Name of TIs	Abbreviation	Formula
1	First Zagreb index	${}^{1}\mathbf{Z}(G)$	$\sum_{\lambda_i\lambda_j\in E(G)} \left(d_{\lambda_i}+d_{\lambda_j}\right)$ [12]
2	general sum-connectivity index	$\chi_\alpha(G)$	$\sum_{\lambda_i\lambda_j\in E(G)} (d_{\lambda_i}+d_{\lambda_j})^{\alpha}$, $\alpha\in\mathbb{R}^{+}$ [13–15]
3	General Randić index	$\mathbf{R}_\alpha(G)$	$\sum_{\lambda_i\lambda_j\in E(G)} \left(d_{\lambda_i}\times d_{\lambda_j}\right)^{\alpha}$, $\alpha\in\mathbb{R}^{+}$
4	Inverse general Randić index	$\mathbf{RR}_\alpha(G)$	$\sum_{\lambda_i\lambda_j\in E(G)} \left(d_{\lambda_i}\times d_{\lambda_j}\right)^{-\alpha}$, $\alpha\in\mathbb{R}^{+}$
5	atom-bond connectivity index	ABC	$\sum_{\lambda_i\lambda_j\in E(G)} \sqrt{\frac{d_{\lambda_i}+d_{\lambda_j}-2}{d_{\lambda_i}\times d_{\lambda_j}}}$ [16]
6	geometric-arithmetic index	GA	$\sum_{\lambda_i\lambda_j\in E(G)} \frac{2\sqrt{d_{\lambda_i}\times d_{\lambda_j}}}{d_{\lambda_i}+d_{\lambda_j}}$ [17]
7	fourth member of the class of *ABC*	ABC_4	$\sum_{\lambda_i\lambda_j\in E(G)} \sqrt{\frac{S_{\lambda_i}+S_v-2}{S_{\lambda_i}\times S_v}}$ [18]
8	fifth version of (GA)	GA_5	$\sum_{\lambda_i\lambda_j\in E(G)} \frac{2\sqrt{S_{\lambda_i}\times S_v}}{S_{\lambda_i}+S_v}$ [**?**]
9	first multiple Zagreb index	$PM_1(G)$,	$\prod_{\lambda_i\lambda_j\in E(G)} (d_{\lambda_i}+d_{\lambda_j})$ [20]
10	second multiple Zagreb index	$PM_2(G)$	$\prod_{\lambda_i\lambda_j\in E(G)} (d_{\lambda_i}\times d_{\lambda_j})$ [20]
11	first Zagreb polynomial	$M_1(G,x)$	$\sum_{\lambda_i\lambda_j\in E(G)} x^{(d_{\lambda_i}+d_{\lambda_j})}$ [20]
12	second Zagreb polynomial	$M_2(G,x)$	$\sum_{\lambda_i\lambda_j\in E(G)} x^{(d_{\lambda_i}\times d_{\lambda_j})}$ [20]

Recently, Koam *et al.* determined the algebraic properties for structures, Magnesium Iodide [21], Fuchsine Acid Dye [22], Backbone DNA Networks [23] and Boron Clusters Sheets [24, 25]. Two-dimensional coronene fractal structures are studies by Khabyah *et al.*

[26] and Hakami *et al.* [27]. Ahmad *et al.* [28] discussed the reverse degree based indices of Fullerene cages networks. Further research studied the topological indices for different graphs like, anti-cancer treatment [29], bioconjugate networks [30], random cyclooctane chains [31], benzenoid systems[32, 33] and Carbon Nanotubes [34]. For further results, see [35–39].

2. RESULTS AND DISCUSSION

Trees, a elementary information creation in computer science and mathematics, find distinct applications across different topics. Trees are ordered arranges consisting of nodes linked by edges, with each node having zero or more child nodes. This flexible data construct is used in frequent fields, including computer science, networking, inheritance, and artificial information. Binary Search Trees (BSTs) enable rapidly probing, insertion, and deletion procedures, making them valuable in records and file techniques. Huffman trees, a alternative of binary trees, are devoted in Huffman coding for lossless information firmness. This performance is applied in file firmness systems like ZIP and JPEG, enhancing storage and conduction competence. Trees show a fundamental role in networking conventions and routing algorithms. In short, trees are flexible and influential data structures with extensive-extending applications in computer science, networking, genetics, artificial intelligence, game theory, and organizational management. Their hierarchical features and able operations make them essential tools for coordinating, analyzing, and operating data in miscellaneous areas. For further detail see [40].

Due to symmetrical collections of nodes at each height in $\diamondsuit-$ary complete trees, enough algebraic possessions can be formulated and their algorithms can be designed and used in different applications[41]. In this article some of those properties are used to formulate some new properties and their proofs are given. In $\diamondsuit-$ary trees accomulated nodes at every particular single level are combine up to $\diamondsuit^{level}$. So $q-$height $\diamondsuit-$ary tree contains the amount of nodes V is determined by implementing the assertion $\frac{\diamondsuit^{\aleph+1}-1}{\diamondsuit-1}$ and the total edges becomes $\frac{\diamondsuit^{\aleph+1}-\diamondsuit}{\diamondsuit-1}$. For understanding the graph of $3-$ary complete tree of 3 height is depicted in Figure 1. Let $e_{d_{\lambda_i},d_{\lambda_j}}$ denote the partition of edges with degree of end nodes. The node partition is shown in Table 4.

TABLE 2. The degree-based node partition of complete $\diamondsuit-$ary Tree

Degree of vertex	Number of vertices
1	$\diamondsuit^{\aleph}$
$\diamondsuit$	1
$\diamondsuit+1$	$\frac{\diamondsuit^{\aleph}-\diamondsuit}{\diamondsuit-1}$
Total	$\frac{\diamondsuit^{\aleph+1}-1}{\diamondsuit-1}$

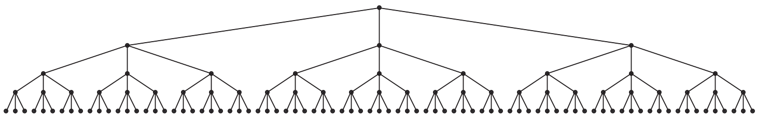

FIGURE 1. A complete 3−ary tree of height 4.

TABLE 3. The degree-based edge partition of complete $\diamondsuit-$ary Tree.

$(d_{\lambda_i}, d_{\lambda_j})$, where $\lambda_i\lambda_j \in E(G)$	Number of edges
$(1, \diamondsuit+1)$	$\diamondsuit^{\aleph}$
$(\diamondsuit, \diamondsuit+1)$	$\diamondsuit$
$(\diamondsuit+1, \diamondsuit+1)$	$\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1}$
Total	$\frac{\diamondsuit^{\aleph+1}-\diamondsuit}{\diamondsuit-1}$

Theorem 2.1. *Let $C_{\diamondsuit,\aleph}$ be a complete $\diamondsuit-$ary trees of height $\aleph$. For $\diamondsuit \geq 2$, $\aleph \geq 2$, then*

(1) $M_\alpha(C_{\diamondsuit,\aleph}) = \diamondsuit^{\aleph} + \diamondsuit^{\alpha} + (\frac{\diamondsuit^{\aleph}-\diamondsuit}{\diamondsuit-1})(\diamondsuit+1)^{\alpha}$

(2) $R_\alpha(C_{\diamondsuit,\aleph}) = (\diamondsuit+1)^{\alpha}(\diamondsuit^{\aleph} + \diamondsuit^{\alpha+1}) + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})(\diamondsuit+1)^{2\alpha}$

(3) $\chi_\alpha(C_{\diamondsuit,\aleph}) = \diamondsuit^{\aleph}(\diamondsuit+2)^{\alpha} + \diamondsuit(2\diamondsuit+1)^{\alpha} + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})(2\diamondsuit+2)^{\alpha}$,
where α is a real number.

Proof. Using the values of Table 3 in the formula of general Randić index that is defined above as:

$$R_\alpha(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} (d_{\lambda_i} d_{\lambda_j})^{\alpha}$$

This implies that

$$\begin{aligned} R_\alpha(C_{\diamondsuit,\aleph}) =& e_{1,\diamondsuit+1}(1\times(\diamondsuit+1))^{\alpha} + e_{\diamondsuit,\diamondsuit+1}(\diamondsuit\times(\diamondsuit+1))^{\alpha} \\ &+ e_{\diamondsuit+1,\diamondsuit+1}((\diamondsuit+1)\times(\diamondsuit+1))^{\alpha} \\ =& (\diamondsuit^{\aleph})(\diamondsuit+1)^{\alpha} + (\diamondsuit)(\diamondsuit(\diamondsuit+1))^{\alpha} + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})(\diamondsuit+1)^{2\alpha} \\ =& (\diamondsuit+1)^{\alpha}(\diamondsuit^{\aleph} + \diamondsuit^{\alpha+1}) + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})(\diamondsuit+1)^{2\alpha}, \end{aligned}$$

and the formula of general sum-connectivity index is

$$\chi_\alpha(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} (d_{\lambda_i} + d_{\lambda_j})^{\alpha}$$

This implies that

$\chi_\alpha(C_{\diamondsuit,\aleph}) = e_{1,\diamondsuit+1}(1+(\diamondsuit+1))^\alpha + e_{\diamondsuit,\diamondsuit+1}(\diamondsuit+(\diamondsuit+1))^\alpha + e_{\diamondsuit+1,\diamondsuit+1}((\diamondsuit+1)+(\diamondsuit+1))^\alpha = \diamondsuit^\aleph(\diamondsuit+2)^\alpha + \diamondsuit(2\diamondsuit+1)^\alpha + (\frac{\diamondsuit^\aleph-\diamondsuit^2}{\diamondsuit-1})(2\diamondsuit+2)^\alpha$. Which completes the proof. □

Theorem 2.2. *For $\diamondsuit \geq 2$, $\aleph \geq 2$, the atom-bound connectivity index ABC of $C_{\diamondsuit,\aleph}$ is given by*

$$ABC(C_{\diamondsuit,\aleph}) = \diamondsuit^\aleph\sqrt{\frac{\diamondsuit}{\diamondsuit+1}} + \diamondsuit\sqrt{\frac{2\diamondsuit-1}{\diamondsuit^2+\diamondsuit}} + \frac{\sqrt{2\diamondsuit}(\diamondsuit^\aleph-\diamondsuit^2)}{(\diamondsuit^2-1)}.$$

Proof. Using the values of Table 3 in the formula of the atom-bond connectivity index that is defined above as:

$$ABC(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} \sqrt{\frac{d_{\lambda_i}+d_{\lambda_j}-2}{d_{\lambda_i}\times d_{\lambda_j}}}$$

This implies that

$ABC(C_{\diamondsuit,\aleph}) = e_{1,\diamondsuit+1}\sqrt{\frac{1+\diamondsuit+1-2}{1\times(\diamondsuit+1)}} + e_{\diamondsuit,\diamondsuit+1}\sqrt{\frac{\diamondsuit+\diamondsuit+1-2}{\diamondsuit\times(\diamondsuit+1)}} + e_{\diamondsuit+1,\diamondsuit+1}\sqrt{\frac{\diamondsuit+1+\diamondsuit+1-2}{(\diamondsuit+1)\times(\diamondsuit+1)}}$.

By using the Table 3, after simplification we get

$ABC(C_{\diamondsuit,\aleph}) = \diamondsuit^\aleph\sqrt{\frac{\diamondsuit}{\diamondsuit+1}} + \diamondsuit\sqrt{\frac{2\diamondsuit-1}{\diamondsuit^2+\diamondsuit}} + \frac{\sqrt{2\diamondsuit}(\diamondsuit^\aleph-\diamondsuit^2)}{(\diamondsuit^2-1)}$.

Which completes the proof. □

Theorem 2.3. *For $\diamondsuit \geq 2$, $\aleph \geq 2$, the geometric-arithmetic index GA of $C_{\diamondsuit,\aleph}$ is given by*

$$GA(C_{\diamondsuit,\aleph}) = 2\diamondsuit^\aleph\frac{\sqrt{\diamondsuit+1}}{\diamondsuit+2} + 2\diamondsuit\frac{\sqrt{\diamondsuit(\diamondsuit+1)}}{2\diamondsuit+1} + \frac{\diamondsuit^\aleph-\diamondsuit^2}{\diamondsuit-1}.$$

Proof. Using the values of Table 3 in the formula of the geometric-arithmetic index that is defined above as:

$$GA(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} \frac{2\sqrt{d_{\lambda_i}\times d_{\lambda_j}}}{d_{\lambda_i}+d_{\lambda_j}}$$

This implies that

$GA(C_{\diamondsuit,\aleph}) = e_{1,\diamondsuit+1}\frac{2\sqrt{1\times(\diamondsuit+1)}}{1+\diamondsuit+1} + e_{\diamondsuit,\diamondsuit+1}\frac{2\sqrt{\diamondsuit\times(\diamondsuit+1)}}{\diamondsuit+\diamondsuit+1} + e_{\diamondsuit+1,\diamondsuit+1}\frac{2\sqrt{(\diamondsuit+1)\times(\diamondsuit+1)}}{\diamondsuit+1+\diamondsuit+1}$.

By using the Table 3, we get

$GA(C_{\diamondsuit,\aleph}) = \diamondsuit^\aleph\frac{2\sqrt{\diamondsuit+1}}{\diamondsuit+2} + \diamondsuit\frac{2\sqrt{\diamondsuit(\diamondsuit+1)}}{2\diamondsuit+1} + (\frac{\diamondsuit^\aleph-\diamondsuit^2}{\diamondsuit-1})\frac{2(\diamondsuit+1)}{2(\diamondsuit+1)}$.

After simplification we obtain

$GA(C_{\diamondsuit,\aleph}) = 2\diamondsuit^\aleph\frac{\sqrt{\diamondsuit+1}}{\diamondsuit+2} + 2\diamondsuit\frac{\sqrt{\diamondsuit(\diamondsuit+1)}}{2\diamondsuit+1} + \frac{\diamondsuit^\aleph-\diamondsuit^2}{\diamondsuit-1}$. Which completes the proof. □

Now, we calculate ABC_4 and GA_5 for complete $\diamondsuit-$ary tree. A complete $\diamondsuit-$ary tree for height $\aleph = 1$ is a star tree with $S_\diamondsuit$ whose degree partitions are already known. For height $h = 2$, two types of edges on degree based sum of neighbors vertices of each edge can be classified. The first edge partition $E_{\diamondsuit+1,2\diamondsuit}$ contains $\diamondsuit^{\aleph}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=\diamondsuit+1$, $S_{\lambda_j}=2\diamondsuit$. The second edge partition $E_{2\diamondsuit,\diamondsuit(\diamondsuit+1)}$ contains $\diamondsuit^{\aleph-1}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=2\diamondsuit$, $S_{\lambda_j}=\diamondsuit(\diamondsuit+1)$. For height $h=3$, three types of edges on degree based sum of neighbors vertices of each edge can be classified. The first edge partition $E_{\diamondsuit+1,2\diamondsuit+1}$ contains $\diamondsuit^{\aleph}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=\diamondsuit+1$, $S_{\lambda_j}=2\diamondsuit+1$. The second edge partition $E_{2\diamondsuit+1,\diamondsuit(\diamondsuit+2)}$ contains $\diamondsuit^{\aleph-1}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=2\diamondsuit+1$, $S_{\lambda_j}=m(m+2)$ and third edge partition $E_{\diamondsuit(\diamondsuit+2),\diamondsuit(\diamondsuit+1)}$ contains $\diamondsuit$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=\diamondsuit(\diamondsuit+2)$, $S_{\lambda_j}=\diamondsuit(\diamondsuit+1)$. Whereas for height $\aleph>3$ these edge partitions are generalized in five types and after calculating these partitions they are used to calculate ABC_4 and GA_5 indices. Table 5 gives such types of edges of the complete $\diamondsuit-$ary trees. The edge set $E(C_{\diamondsuit,\aleph})$ divided into four edge partitions based on degree of end vertices. The first edge partition $E_{\diamondsuit+1,2\diamondsuit+1}(C_{\diamondsuit,\aleph})$ contains $\diamondsuit^{\aleph}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=\diamondsuit+1$, $S_{\lambda_j}=2\diamondsuit+1$ and $e_{\diamondsuit+1,2\diamondsuit+1}=|E_{\diamondsuit+1,2\diamondsuit+1}(C_{\diamondsuit,\aleph})|$. The second edge partition $E_{2\diamondsuit+1,(\diamondsuit+1)^2}(C_{\diamondsuit,\aleph})$ contains $\diamondsuit^{\aleph-1}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=2\diamondsuit+1$, $S_{\lambda_j}=(\diamondsuit+1)^2$ and $e_{2\diamondsuit+1,(\diamondsuit+1)^2}=|E_{2\diamondsuit+1,(\diamondsuit+1)^2}(C_{\diamondsuit,\aleph})|$. The third edge partition $E_{\diamondsuit(\diamondsuit+1),\diamondsuit(\diamondsuit+2)}(C_{\diamondsuit,\aleph})$ contains $\diamondsuit$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=\diamondsuit(\diamondsuit+1)$, $S_{\lambda_j}=\diamondsuit(\diamondsuit+2)$ and $e_{\diamondsuit(\diamondsuit+1),\diamondsuit(\diamondsuit+2)}=|E_{\diamondsuit(\diamondsuit+1),\diamondsuit(\diamondsuit+2)}(C_{\diamondsuit,\aleph})|$. The fourth edge partition $E_{\diamondsuit(\diamondsuit+2),(\diamondsuit+1)^2}(C_{\diamondsuit,\aleph})$ contains $\diamondsuit^2$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=\diamondsuit(\diamondsuit+2)$, $S_{\lambda_j}=(\diamondsuit+1)^2$ and $e_{\diamondsuit(\diamondsuit+2),(\diamondsuit+1)^2}=|E_{\diamondsuit(\diamondsuit+2),(\diamondsuit+1)^2}(C_{\diamondsuit,\aleph})|$. The fifth edge partition $E_{(\diamondsuit+1)^2,(\diamondsuit+1)^2}(C_{\diamondsuit,\aleph})$ contains $\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{\diamondsuit-1}$ edges $\lambda_i\lambda_j$, where $S_{\lambda_i}=S_{\lambda_j}=(\diamondsuit+1)^2$ and $e_{(\diamondsuit+1)^2,(\diamondsuit+1)^2}=|E_{(\diamondsuit+1)^2,(\diamondsuit+1)^2}(C_{\diamondsuit,\aleph})|$.

TABLE 4. The vertex partition of graph $C_{\diamondsuit,\aleph}$ based on degree sum of neighbor vertices of end vertices of each edge.

degree sum of neighbor vertices of end vertices of each edge	Number of vertices
$\diamondsuit+1$	$\diamondsuit^{\aleph}$
$2\diamondsuit+1$	$\diamondsuit^{\aleph-1}$
$\diamondsuit(\diamondsuit+1)$	1
$\diamondsuit(\diamondsuit+2)$	$\diamondsuit$
$(\diamondsuit+1)^2$	$\frac{\diamondsuit^{\aleph-1}-\diamondsuit^2}{\diamondsuit-1}$
Total	$\frac{\diamondsuit^{\aleph+1}-1}{\diamondsuit-1}$

Theorem 2.4. *For $\diamondsuit\geq 2$, $\aleph>3$, the 4th atom-bound connectivity index ABC_4 of $C_{\diamondsuit,\aleph}$ is given by*

$$ABC_4(C_{\diamondsuit,\aleph})=(\diamondsuit^{\aleph})\sqrt{\frac{3\diamondsuit}{2\diamondsuit^2+3\diamondsuit+1}}+\frac{\diamondsuit^{\aleph-1}}{\diamondsuit+1}\sqrt{\frac{\diamondsuit^2+4\diamondsuit}{2\diamondsuit+1}}+\sqrt{\frac{2\diamondsuit^2+3\diamondsuit-2}{\diamondsuit^2+3\diamondsuit+2}}+\frac{\diamondsuit^2}{\diamondsuit+1}\sqrt{\frac{2\diamondsuit^2+4\diamondsuit-1}{\diamondsuit^2+2\diamondsuit}}+\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{(\diamondsuit-1)(\diamondsuit+1)^2}\sqrt{2\diamondsuit^2+4m}.$$

TABLE 5. Based on the sum-degree of neighbor vertices, the edge partition of $C_{\diamondsuit,\aleph}$ graph.

$(S_{\lambda_i}, S_{\lambda_j})$, where $\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})$	Number of edges
$(\diamondsuit+1, 2\diamondsuit+1)$	$\diamondsuit^{\aleph}$
$(2\diamondsuit+1, (\diamondsuit+1)^2)$	$\diamondsuit^{\aleph-1}$
$(\diamondsuit(\diamondsuit+1), \diamondsuit(\diamondsuit+2))$	$\diamondsuit$
$(\diamondsuit(\diamondsuit+2), (\diamondsuit+1)^2)$	$\diamondsuit^2$
$((\diamondsuit+1)^2, (\diamondsuit+1)^2)$	$\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{\diamondsuit-1}$
Total	$\frac{\diamondsuit^{\aleph+1}-\diamondsuit}{\diamondsuit-1}$

Proof. It is easy to see that edges has five types $e_{\diamondsuit+1,2\diamondsuit+1}$, $e_{2\diamondsuit+1,(\diamondsuit+1)^2}$, $e_{\diamondsuit(\diamondsuit+1),\diamondsuit(\diamondsuit+2)}$, $e_{\diamondsuit(\diamondsuit+2),(\diamondsuit+1)^2}$ and $e_{(\diamondsuit+1)^2,(\diamondsuit+1)^2}$ that are shown in Table 5. The ABC_4 is defined as:

$$ABC_4(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} \sqrt{\frac{S_{\lambda_i}+S_{\lambda_j}-2}{S_{\lambda_i}\times S_{\lambda_j}}}$$

This implies that

$$\begin{aligned}
ABC_4(C_{\diamondsuit,\aleph}) =& e_{\diamondsuit+1,2\diamondsuit+1}\sqrt{\frac{\diamondsuit+1+2\diamondsuit+1-2}{(\diamondsuit+1)\times(2\diamondsuit+1)}} + e_{2\diamondsuit+1,(\diamondsuit+1)^2}\sqrt{\frac{(2\diamondsuit+1)+(\diamondsuit+1)^2-2}{(2\diamondsuit+1)\times(\diamondsuit+1)^2}} \\
&+ e_{\diamondsuit(\diamondsuit+1),\diamondsuit(\diamondsuit+2)}\sqrt{\frac{\diamondsuit(\diamondsuit+1)+\diamondsuit(\diamondsuit+2)-2}{\diamondsuit(\diamondsuit+1)\times\diamondsuit(\diamondsuit+2)}} + e_{\diamondsuit(\diamondsuit+2),(\diamondsuit+1)^2}\sqrt{\frac{\diamondsuit(\diamondsuit+2)+(\diamondsuit+1)^2-2}{\diamondsuit(\diamondsuit+2)\times(\diamondsuit+1)^2}} \\
&+ e_{(\diamondsuit+1)^2,(\diamondsuit+1)^2}\sqrt{\frac{(\diamondsuit+1)^2+(\diamondsuit+1)^2-2}{(\diamondsuit+1)^2\times(\diamondsuit+1)^2}}.
\end{aligned}$$

By using the Table 5, we get

$$\begin{aligned}
ABC_4(C_{\diamondsuit,\aleph}) =& (\diamondsuit^{\aleph})\sqrt{\frac{3\diamondsuit}{2\diamondsuit^2+3\diamondsuit+1}} + (\diamondsuit^{\aleph-1})\frac{1}{\diamondsuit+1}\sqrt{\frac{\diamondsuit^2+4\diamondsuit}{2\diamondsuit+1}} + (\diamondsuit)\frac{1}{\diamondsuit}\sqrt{\frac{2\diamondsuit^2+3\diamondsuit-2}{\diamondsuit^2+3\diamondsuit+2}} \\
&+ (\diamondsuit^2)\frac{1}{\diamondsuit+1}\sqrt{\frac{2\diamondsuit^2+4\diamondsuit-1}{\diamondsuit^2+2\diamondsuit}} + (\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{\diamondsuit-1})\frac{1}{(\diamondsuit+1)^2}\sqrt{2\diamondsuit^2+4\diamondsuit} \\
=& (\diamondsuit^{\aleph})\sqrt{\frac{3\diamondsuit}{2\diamondsuit^2+3\diamondsuit+1}} + \frac{\diamondsuit^{\aleph-1}}{\diamondsuit+1}\sqrt{\frac{\diamondsuit^2+4\diamondsuit}{2\diamondsuit+1}} + \sqrt{\frac{2\diamondsuit^2+3\diamondsuit-2}{\diamondsuit^2+3\diamondsuit+2}} \\
&+ \frac{\diamondsuit^2}{\diamondsuit+1}\sqrt{\frac{2\diamondsuit^2+4\diamondsuit-1}{\diamondsuit^2+2\diamondsuit}} + \frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{(\diamondsuit-1)(\diamondsuit+1)^2}\sqrt{2\diamondsuit^2+4\diamondsuit}.
\end{aligned}$$

That completes the proof. □

Theorem 2.5. *For $\diamondsuit \geq 2$, $\aleph > 3$, the 5th geometric-arithmetic index GA_5 of $C_{\diamondsuit,\aleph}$ is given by* $GA_5(C_{\diamondsuit,\aleph}) = 2\diamondsuit^{\aleph}\frac{\sqrt{2\diamondsuit^2+3\diamondsuit+1}}{3\diamondsuit+2} + 2\diamondsuit^{\aleph-1}\frac{\sqrt{2\diamondsuit^3+5\diamondsuit^2+4\diamondsuit+1}}{\diamondsuit^2+4\diamondsuit+2} + 2\diamondsuit\frac{\sqrt{\diamondsuit^2+3\diamondsuit+2}}{2\diamondsuit+3}$
$+2\diamondsuit^2(\diamondsuit+1)\frac{\sqrt{\diamondsuit^2+2\diamondsuit}}{2\diamondsuit^2+4\diamondsuit+1} + (\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{\diamondsuit-1})$.

Proof. The fifth geometric-arithmetic index GA_5 is defined as:

$$GA_5(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j\in E(C_{\diamondsuit,\aleph})} \frac{2\sqrt{S_{\lambda_i}\times S_{\lambda_j}}}{S_{\lambda_i}+S_{\lambda_j}}$$

This implies that

$$GA_5(C_{\diamondsuit,\aleph}) = e_{\diamondsuit+1,2\diamondsuit+1}\frac{2\sqrt{(\diamondsuit+1)\times(2\diamondsuit+1)}}{\diamondsuit+1+2\diamondsuit+1} + e_{2\diamondsuit+1,(\diamondsuit+1)^2}\frac{2\sqrt{(2\diamondsuit+1)\times(\diamondsuit+1)^2}}{2\diamondsuit+1+(\diamondsuit+1)^2}$$

$+e_{\diamondsuit(\diamondsuit+1),\diamondsuit(\diamondsuit+2)}\frac{2\sqrt{\diamondsuit(\diamondsuit+1)\times\diamondsuit(\diamondsuit+2)}}{\diamondsuit(\diamondsuit+1)+\diamondsuit(\diamondsuit+2)} + e_{\diamondsuit(\diamondsuit+2),(\diamondsuit+1)^2}\frac{2\sqrt{\diamondsuit(\diamondsuit+2)\times(\diamondsuit+1)^2}}{\diamondsuit(\diamondsuit+2)+(\diamondsuit+1)^2}$
$+e_{(\diamondsuit+1)^2,(\diamondsuit+1)^2}\frac{2\sqrt{(\diamondsuit+1)^2\times(\diamondsuit+1)^2}}{(\diamondsuit+1)^2+(\diamondsuit+1)^2}$. By using the Table 5, we get
$GA_5(C_{\diamondsuit,\aleph}) = (\diamondsuit^{\aleph})\frac{2\sqrt{2\diamondsuit^2+3\diamondsuit+1}}{3\diamondsuit+2} + (\diamondsuit^{\aleph-1})\frac{2\sqrt{2\diamondsuit^3+5\diamondsuit^2+4\diamondsuit+1}}{\diamondsuit^2+4\diamondsuit+2} + (\diamondsuit)\frac{2\sqrt{\diamondsuit^2(\diamondsuit^2+3\diamondsuit+2)}}{\diamondsuit(2\diamondsuit+3)} +$
$(\diamondsuit^2)\frac{2\sqrt{\diamondsuit(\diamondsuit+2)\times(\diamondsuit+1)^2}}{2\diamondsuit^2+4\diamondsuit+1} + (\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{\diamondsuit-1})\frac{2(\diamondsuit+1)^2}{2(\diamondsuit+1)^2}$.
After easy simplification we obtain
$GA_5(C_{\diamondsuit,\aleph}) = 2\diamondsuit^{\aleph}\frac{\sqrt{2\diamondsuit^2+3\diamondsuit+1}}{3\diamondsuit+2} + 2\diamondsuit^{\aleph-1}\frac{\sqrt{2\diamondsuit^3+5\diamondsuit^2+4\diamondsuit+1}}{\diamondsuit^2+4\diamondsuit+2} + 2\diamondsuit\frac{\sqrt{\diamondsuit^2+3\diamondsuit+2}}{2\diamondsuit+3}$
$+2\diamondsuit^2(\diamondsuit+1)\frac{\sqrt{\diamondsuit^2+2\diamondsuit}}{2\diamondsuit^2+4\diamondsuit+1} + (\frac{\diamondsuit^{\aleph-1}-\diamondsuit^3}{\diamondsuit-1})$.

□

Theorem 2.6. *For $\diamondsuit \geq 2$, $\aleph \geq 2$, then*

(1) $HM(C_{\diamondsuit,\aleph}) = \diamondsuit^{\aleph}(\diamondsuit+2)^2 + \diamondsuit(2\diamondsuit+1)^2 + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})(2\diamondsuit+2)^2$
(2) $PM_1(C_{\diamondsuit,\aleph}) = (\diamondsuit+2)^{\diamondsuit^{\aleph}} \times (2\diamondsuit+1)^{\diamondsuit} \times (2\diamondsuit+2)^{\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1}}$
(3) $PM_2(C_{\diamondsuit,\aleph}) = \diamondsuit^{\diamondsuit}(\diamondsuit+1)^{\frac{(\diamondsuit+1)(\diamondsuit^{\aleph}-\diamondsuit)}{\diamondsuit-1}}$
(4) $M_1(C_{\diamondsuit,\aleph},x) = \diamondsuit^{\aleph}x^{\diamondsuit+2} + \diamondsuit x^{2\diamondsuit+1} + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})x^{2\diamondsuit+2}$
(5) $M_2(C_{\diamondsuit,\aleph},x) = \diamondsuit^{\aleph}x^{\diamondsuit+1} + \diamondsuit x^{\diamondsuit(\diamondsuit+1)} + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})x^{(\diamondsuit+1)^2}$

Proof. Since,

$$HM(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j\in E(C_{\diamondsuit,\aleph})} \left(d_{\lambda_i}+d_{\lambda_j}\right)^2$$

$HM(C_{\diamondsuit,\aleph}) = \sum_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} \left[d_{\lambda_i} + d_{\lambda_j}\right]^2 + \sum_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} \left[d_{\lambda_i} + d_{\lambda_j}\right]^2$
$+ \sum_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} \left[d_{\lambda_i} + d_{\lambda_j}\right]^2$

$HM(C_{\diamondsuit,\aleph}) = e_{1,\diamondsuit+1}(1+(\diamondsuit+1))^2 + e_{\diamondsuit,\diamondsuit+1}(\diamondsuit+(\diamondsuit+1))^2 + e_{\diamondsuit+1,\diamondsuit+1}((\diamondsuit+1)+(\diamondsuit+1))^2$

After putting the values of edge partitions, we get

$HM(C_{\diamondsuit,\aleph}) = \diamondsuit^{\aleph}(\diamondsuit+2)^2 + \diamondsuit(2\diamondsuit+1)^2 + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1})(2\diamondsuit+2)^2$. Since,

$$PM_1(C_{\diamondsuit,\aleph}) = \prod_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} (d_{\lambda_i} + d_{\lambda_j})$$

$PM_1(C_{\diamondsuit,\aleph}) = \prod_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} (d_{\lambda_i} + d_{\lambda_j}) \times \prod_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} (d_{\lambda_i} + d_{\lambda_j}) \times \prod_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} (d_{\lambda_i} + d_{\lambda_j})$

$PM_1(C_{\diamondsuit,\aleph}) = (\diamondsuit+2)^{|E_1(C_{\diamondsuit,\aleph})|} \times (2\diamondsuit+1)^{|E_2(C_{\diamondsuit,\aleph})|} \times (2\diamondsuit+2)^{|E_3(C_{\diamondsuit,\aleph})|}$

$PM_1(C_{\diamondsuit,\aleph}) = (\diamondsuit+2)^{\diamondsuit^{\aleph}} \times (2\diamondsuit+1)^{\diamondsuit} \times (2\diamondsuit+2)^{\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1}}$. Now, since

$$PM_2(C_{\diamondsuit,\aleph}) = \prod_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} (d_{\lambda_i} \times d_{\lambda_j})$$

$PM_2(C_{\diamondsuit,\aleph}) = \prod_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} (d_{\lambda_i} \times d_{\lambda_j}) \times \prod_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} (d_{\lambda_i} \times d_{\lambda_j}) \times \prod_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} (d_{\lambda_i} \times d_{\lambda_j})$

$PM_2(C_{\diamondsuit,\aleph}) = (\diamondsuit+1)^{|E_1(C_{\diamondsuit,\aleph})|} \times (\diamondsuit(\diamondsuit+1))^{|E_2(C_{\diamondsuit,\aleph})|} \times ((\diamondsuit+1)^2)^{|E_3(C_{\diamondsuit,\aleph})|}$

$= (\diamondsuit+1)^{\diamondsuit^{\aleph}} \times (\diamondsuit(\diamondsuit+1))^{\diamondsuit} \times ((\diamondsuit+1)^2)^{\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1}}$. After simplification we get

$PM_2(C_{\diamondsuit,\aleph}) = \diamondsuit^{\diamondsuit}(\diamondsuit+1)^{\frac{(\diamondsuit+1)(\diamondsuit^{\aleph}-\diamondsuit)}{\diamondsuit-1}}$. As,

$$M_1(C_{\diamondsuit,\aleph}, x) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} + d_{\lambda_j})}$$

$M_1(C_{\diamondsuit,\aleph}, x) = \sum_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} + d_{\lambda_j})} + \sum_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} + d_{\lambda_j})} + \sum_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} + d_{\lambda_j})}$

$= \sum_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} x^{\diamondsuit+2} + \sum_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} x^{2\diamondsuit+1} + \sum_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} x^{2\diamondsuit+2}$

$= |E_1(C_{\diamondsuit,\aleph})| x^{\diamondsuit+2} + |E_2(C_{\diamondsuit,\aleph})| x^{2\diamondsuit+1} + |E_3(C_{\diamondsuit,\aleph})| x^{2\diamondsuit+2}$

$= \diamondsuit^{\aleph} x^{\diamondsuit+2} + \diamondsuit x^{2\diamondsuit+1} + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1}) x^{2\diamondsuit+2}$, As

$$M_2(C_{\diamondsuit,\aleph}, x) = \sum_{\lambda_i\lambda_j \in E(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} \times d_{\lambda_j})}$$

$M_2(C_{\diamondsuit,\aleph}, x) = \sum_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} \times d_{\lambda_j})} + \sum_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} \times d_{\lambda_j})} + \sum_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} x^{(d_{\lambda_i} \times d_{\lambda_j})}$

$= \sum_{\lambda_i\lambda_j \in E_1(C_{\diamondsuit,\aleph})} x^{\diamondsuit+1} + \sum_{\lambda_i\lambda_j \in E_2(C_{\diamondsuit,\aleph})} x^{\diamondsuit(\diamondsuit+1)} + \sum_{\lambda_i\lambda_j \in E_3(C_{\diamondsuit,\aleph})} x^{(\diamondsuit+1)^2}$. By putting the values, we obtain $M_2(C_{\diamondsuit,\aleph},x) = \diamondsuit^{\aleph} x^{\diamondsuit+1} + \diamondsuit x^{\diamondsuit(\diamondsuit+1)} + (\frac{\diamondsuit^{\aleph}-\diamondsuit^2}{\diamondsuit-1}) x^{(\diamondsuit+1)^2}$.

□

3. Conclusion

This paper deal with the degree-based indices for complete $\diamondsuit$-ary tree. Its numerical and graphical representation are shown in Table 6 and Figure 2. The degree-based indices characterize a elementary appearance of chemical graph theory, supporting appreciated intuitions into the connectivity and structural assets of molecules. By counting the degree of splitting, equilibrium, and density within molecular graphs, these indices provide as crucial tools in molecular modeling, quantitative structure-activity relationship (QSAR) studies, and drug layout. Examples such as the Wiener index, Randić index, and Zagreb indices extend resourceful means of indicating molecular analysis situs and expecting various physicochemical belongings. As computational modes continue to onslaught, degree-based indices will persist connected to understanding molecular structures and managing molecular research in disciplines such as medications, materials science, and computational natural science.

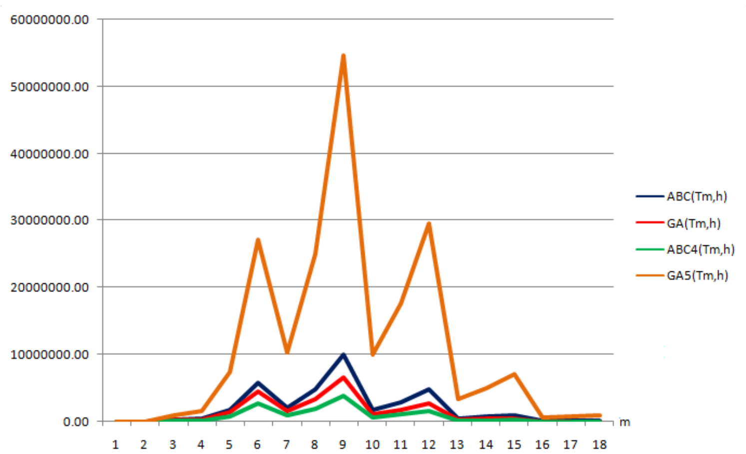

FIGURE 2. Shows comparison between ABC, GA, ABC_4, GA_5 against the value of $\diamondsuit$.

TABLE 6. Shows comparison between ABC, GA, ABC_4, GA_5 against the value of $\diamondsuit$.

Inputs		Theorem 2.2	Theorem 2.3	Theorem 2.4	Theorem 2.5
$\diamondsuit$	h	$ABC(C_{\diamondsuit,\aleph})$	$GA(C_{\diamondsuit,\aleph})$	$ABC_4(C_{\diamondsuit,\aleph})$	$GA_5(C_{\diamondsuit,\aleph})$
2	9	758.13	953.36	569.06	1901.47
3	9	23071.82	25586.37	15122.84	72123.22
4	9	283898.42	282770.58	166139.46	1019635.89
5	8	408059.01	371035.53	217098.08	1625036.63
6	8	1721260.63	1446883.54	844117.63	7449384.51
7	8	5841850.24	4584203.32	2668898.95	27118711.31
8	7	2110365.96	1557883.19	905660.31	10410608.50
9	7	4791177.66	3347883.81	1944253.83	24941052.59
10	7	9986356.44	6638817.97	3852701.87	54556549.66
11	6	1765384.83	1121288.42	650412.10	10076726.77
12	6	2971135.93	1809468.91	1049293.92	17654313.38
13	6	4797729.18	2810268.40	1629403.07	29585090.78
14	5	534181.39	301742.92	174944.33	3409503.95
15	5	753829.56	411592.93	238641.91	4968804.84
16	5	1040529.20	550280.62	319085.30	7068796.72
17	4	82858.55	42518.95	24659.26	579135.17
18	4	104125.89	51931.97	30124.02	747610.36
19	4	129252.48	62745.02	36404.21	951971.59

COMPETING INTERESTS

The authors declare that they have no competing interests.

ACKNOWLEDGEMENTS

The authors are grateful to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

AUTHOR'S CONTRIBUTIONS

All authors equally contributed to this work. All authors read and approved the final manuscript.

REFERENCES

[1] Lowel, H., Lemont, K. (1976). *Molecular connectivity in chemistry and drug reserach*, Boston Academic Press.

[2] Wiener, H. (1947). Structural determination of paraffin boiling points. *Journal of the American chemical society*, **69**(1), 17-20.
[3] Dobrynin, A. A., Entringer, R., and Gutman, I. (2001). Wiener index of trees: theory and applications. *Acta Applicandae Mathematica*, **66**, 211-249.
[4] Gutman, I., and Polansky, O. E. (2012). *Mathematical concepts in organic chemistry*. Springer Science and Business Media.
[5] Randic, M. (1975). Characterization of molecular branching. *Journal of the American Chemical Society*, **97**(23), 6609-6615.
[6] Balaban, A. T. (1982). Highly discriminating distance-based topological index. *Chemical physics letters*, **89**(5), 399-404.
[7] Du, Z., Zhou, B., and Trinajstic, N. (2011). On the general sum-connectivity index of trees. *Applied Mathematics Letters*, **24**(3), 402-405.
[8] Lu, M., and Gao, J. (2007). On the Randic index of quasi-tree graphs. *Journal of Mathematical Chemistry*, **42**(3), 297-310.
[9] Xing, R., Zhou, B., and Trinajstic, N. (2010). Sum-connectivity index of molecular trees. *Journal of mathematical chemistry*, **48**, 583-591.
[10] Zaman, S., Koam, A. N., Al Khabyah, A., and Ahmad, A. (2022). The Kemeny's Constant and Spanning Trees of Hexagonal Ring Network. *Computers, Materials and Continua*, **73**(3).
[11] Ahmad, A., and Koam, A. N. (2020). Computing the topological descriptors of line graph of the complete m-ary trees. *Journal of Intelligent and Fuzzy Systems*, **39**(1), 1081-1088.
[12] Li, X., and Zhao, H. (2004). Trees with the first three smallest and largest generalized topological indices. *MATCH Commun. Math. Comput. Chem*, **50**, 57-62.
[13] Zhou, B., and Trinajstić, N. (2009). On a novel connectivity index. *Journal of mathematical chemistry*. **46**, 1252-1270.
[14] Zhou, B., Trinajstić, N., (2010). On general sum-connectivity index. *J. Math. Chem.*, **47**, 210-218.
[15] Shirdel, G. H., Rezapour, H., and Sayadi, A. M. (2013). The hyper-Zagreb index of graph operations. *Iranian Journal of Mathematical Chemistry* **4**(2) 213-220.
[16] Estrada, E., Torres, L., Rodriguez, L., and Gutman, I. (1998). *An atom-bond connectivity index: modelling the enthalpy of formation of alkanes*. NISCAIR-CSIR, India.
[17] Vukičević, D., and Furtula, B. (2009). Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges. *Journal of mathematical chemistry*, **46**, 1369-1376.
[18] Ghorbani, M., and Hosseinzadeh, M. A. (2010). Computing ABC4 index of nanostar dendrimers. *Optoelectronics and Advanced Materials-Rapid Communications*, **4**, 1419-1422.

[19] Graovac, A., Ghorbani, M., and Hosseinzadeh, M. A. (2011). Computing fifth geometric-arithmetic index for nanostar dendrimers. *Journal of Discrete Mathematics and Its Applications*, **1**(1-2), 33-42.

[20] Ghorbani, M., and Azimi, N. (2012). Note on multiple Zagreb indices. *Iranian journal of mathematical chemistry*, **3**(2), 137-143.

[21] Koam, A. N., Ahmad, A., Azeem, M., and Siddiqui, M. K. (2023). Algebraic properties for molecular structure of magnesium iodide. *CMES-Comput. Model. Eng. Sci*, **135**(2), 1131-1146.

[22] Koam, A. N., Ahmad, A., Zaman, S., Masmali, I., and Ghazwani, H. (2024). Fundamental aspects of the molecular topology of fuchsine acid dye with connection numbers. *The European Physical Journal E*, **47**(4), 24.

[23] Ahmad, A., Koam, A. N., Masmali, I., Azeem, M., and Ghazwani, H. (2023). Connection number topological aspect for backbone DNA networks. *The European Physical Journal E*, **46**(12), 120.

[24] Koam, A. N., Ahmad, A., and Azeem, M. (2023). Boron clusters sheets with algebraic properties. *Molecular Physics*, **121**(21), e2228424.

[25] Refaee, E. A., Ahmad, A., and Azeem, M. (2023). Sombor indices of ?-sheet of boron clusters. *Molecular Physics*, **121**(15), e2214953.

[26] Khabyah, A., Ahmad, A., Azeem, M., Ahmad, Y., and Koam, A. N. (2023). Reverse-degree-based topological indices of two-dimensional coronene fractal structures. *Physica Scripta*, **99(1)**, 015216.

[27] Hakami, K. H., Ahmad, A., Azeem, M., Husain, S., and Koam, A. N. (2023). A study of two-dimensional coronene fractal structures with M-polynomials. *International Journal of Quantum Chemistry*, **123**(13), e27112.

[28] Ahmad, A., Koam, A. N., and Azeem, M. (2023). Reverse-degree-based topological indices of fullerene cage networks. *Molecular Physics*, **121**(14), e2212533.

[29] Zhang, X., Bajwa, Z. S., Zaman, S., Munawar, S., and Li, D. (2024). The study of curve fitting models to analyze some degree-based topological indices of certain anti-cancer treatment. *Chemical Papers*, **78**(2), 1055-1068.

[30] Ullah, A., Zaman, S., Hamraz, A., and Muzammal, M. (2023). On the construction of some bioconjugate networks and their structural modeling via irregularity topological indices. *The European Physical Journal E*, **46**(8), 72.

[31] Yousaf, S., Iqbal, Z., Tariq, S., Aslam, A., Tchier, F., and Issa, A. (2024). Computation of expected values of some connectivity based topological descriptors of random cyclooctane chains. *Scientific Reports*, **14**(1), 7713.

[32] Ahmad, A. (2018). On the degree based topological indices of benzene ring embedded in P-type-surface in 2D network. *Hacettepe Journal of Mathematics and Statistics*, **47**(1), 9-18.

[33] Raza, Z. (2022). Zagreb connection indices for some benzenoid systems. *Polycyclic Aromatic Compounds*, **42**(4), 1814-1827.

[34] Nadeem, M. F., Azeem, M., and Farman, I. (2022). Comparative study of topological indices for capped and uncapped carbon nanotubes. *Polycyclic Aromatic Compounds*, **42**(7), 4666-4683.

[35] Hayat, S., and Imran, M. (2015). Computation of certain topological indices of nanotubes covered by C 5 and C 7. *Journal of Computational and Theoretical Nanoscience*, **12**(4), 533-541.

[36] Hayat, S., and Imran, M. (2015). Computation of certain topological indices of nanotubes covered by C 5 and C 7. *Journal of Computational and Theoretical Nanoscience*, **12**(4), 533-541.

[37] Nadeem, M. F., Zafar, S., and Zahid, Z. (2015). On certain topological indices of the line graph of subdivision graphs. *Applied mathematics and computation*, **271**, 790-794.

[38] Bača, M., Horváthová, J., Mokrišová, M., and Suhányiová, A. (2015). On topological indices of fullerenes. *Applied Mathematics and Computation*, **251**, 154-161.

[39] Sector, H. (2015). Computing topological polynomials of certain nanostructures. *Journal of Optoelectronics and Advanced Materials*, **17**(5-6), 877-883.

[40] Edition, S., and Rosen, K. H. *Discrete Mathematics and Its Applications*.

[41] Ahmad, A., Asim, M. A., Bača, M., and Hasni, R. (2018). Computing edge irregularity strength of complete m-ary trees using algorithmic approach. *UPB Sci. Bull., Ser. A*, **80**(3), 145-152.