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ON PARAMETRIZED FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES

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Abstract. Through this research article we prove parameterized inequality of fractional Hermite-Hadamard type. So, we acquired many results of Hermite-Hadamard type of inequalities.
Keywords: Convex function, Hermite-Hadamard inequality, Riemann-Liouville, Fractional integral.

1. Introduction

Let $\sigma : I \to \mathbb{R}$ be a convex function defined on I such that $\rho_1, \rho_2 \in I$ with $\rho_1 < \rho_2$. Then the following HH inequality [8] holds

$$\sigma\left(\frac{\rho_1+\rho_2}{2}\right) \leq \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\sigma(x)dx \leq \frac{\sigma(\rho_1)+\sigma(\rho_2)}{2}. \tag{1}$$

Many endeavours have been gone on generalizations extensions and variants of Heremite Hadamard (HH) type inequalities for the past 60 years (see [2–5, 10, 15–24]).
For more results related to HH inequality we recommend [1, 9, 13, 14, 26–29, 32, 33].
In the following, we give the definition of fractional Riemann-Liouville integral, which will be used in the later part of the paper. For more detail, one can consult [7, 31].

Definition 1.1. *Let $\sigma \in L[\rho_1, \rho_2]$. The left-sided and right-sided Riemann-Liouville fractional integrals $J^{\zeta}_{\rho_1^+}\sigma$ and $J^{\zeta}_{\rho_2^-}\sigma$ of order $\zeta > 0$ with $\rho_1 \geq 0$ are defined by*

$$J^{\zeta}_{\rho_1^+}\sigma(x) = \frac{1}{\Gamma(\zeta)}\int_{\rho_1}^{x}(x-t)^{\zeta-1}\sigma(t)dt, \quad with \quad x > \rho_1$$

and

$$J^{\zeta}_{\rho_2^-}\sigma(x) = \frac{1}{\Gamma(\zeta)}\int_{x}^{\rho_2}(t-x)^{\zeta-1}\sigma(t)dt, \quad with \quad x < \rho_2$$

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respectively. where, $\Gamma(\zeta)$ is the Gamma function and its definition is

$$\Gamma(\zeta)=\int_0^\infty e^{-u}u^{\zeta-1}du. \tag{2}$$

It is to be noted that $J^0_{\rho_1^+}\sigma(x)=J^0_{\rho_2^-}\sigma(x)=\sigma(x)$. In the case of $\zeta=1$, the fractional integral reduces to the classical integral.

In 2010, Havva Kavurmaci *et. al*. have obtained the following result:

Lemma 1.2 ([11]). *Let $\rho_1,\rho_2\in I^\circ$(interior of I) with $\rho_1<\rho_2$ and $\sigma:I^\circ\to\mathbb{R}$ be a differentiable function on I°. Then the equality*

$$\begin{aligned}&\frac{(x-\rho_1)\sigma(\rho_1)+(\rho_2-x)\sigma(\rho_2)}{\rho_2-\rho_1}-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\sigma(x)dx\\ =\;&\frac{(x-\rho_1)^2}{\rho_2-\rho_1}\int_0^1(t-1)\sigma'(tx+(1-t)\rho_1)dt+\frac{(\rho_2-x)^2}{\rho_2-\rho_1}\int_0^1(1-t)\sigma'(tx+(1-t)\rho_2)dt.\end{aligned} \tag{3}$$

holds for any $x\in[\rho_1,\rho_2]$ if $\sigma'\in L[\rho_1,\rho_2]$.

Theorem 1.3 ([11]). *Let $\rho_1,\rho_2\in I^\circ$(interior of I) with $\rho_1<\rho_2$ and $\sigma:I^\circ\to\mathbb{R}$ be a differentiable function on I°. Then the inequality*

$$\begin{aligned}&\left|\frac{(x-\rho_1)\sigma(\rho_1)+(\rho_2-x)\sigma(\rho_2)}{\rho_2-\rho_1}-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\sigma(x)dx\right|\\ \le\;&\frac{(x-\rho_1)^2}{\rho_2-\rho_1}\left[\frac{|\sigma'(x)|+2|\sigma'(\rho_1)|}{6}\right]+\frac{(\rho_2-x)^2}{\rho_2-\rho_1}\left[\frac{|\sigma'(x)|+2|\sigma'(\rho_2)|}{6}\right].\end{aligned}$$

holds for any $x\in[\rho_1,\rho_2]$ if $\sigma'\in L[\rho_1,\rho_2]$ and $|\sigma'|$ is convex.

Theorem 1.4 ([11]). *Let $\rho_1,\rho_2\in I^\circ$(interior of I) with $\rho_1<\rho_2$, $p,q>1$ such that $p^{-1}+q^{-1}=1$ and $\sigma:I^\circ\to\mathbb{R}$ be a differentiable function on I°. Then the inequality*

$$\begin{aligned}&\left|\frac{(x-\rho_1)\sigma(\rho_1)+(\rho_2-x)\sigma(\rho_2)}{\rho_2-\rho_1}-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\sigma(x)dx\right|\\ \le\;&\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\frac{(x-\rho_1)^2\left[|\sigma'(x)|^q+2|\sigma'(\rho_1)|^q\right]^{\frac{1}{q}}+(\rho_2-x)^2\left[|\sigma'(x)|^q+2|\sigma'(\rho_2)|^q\right]^{\frac{1}{q}}}{\rho_2-\rho_1}\right].\end{aligned}$$

holds for any $x\in[\rho_1,\rho_2]$ if $\sigma'\in L[\rho_1,\rho_2]$ and $|\sigma'|^q$ is convex.

Theorem 1.5 ([11]). *Let $\rho_1,\rho_2 \in I^\circ$(interior of I) with $\rho_1<\rho_2$, $q>1$ and $\sigma : I^\circ \to \mathbb{R}$ be a differentiable function on I°. Then the inequality*

$$\begin{aligned}&\left|\frac{(x-\rho_1)\sigma(\rho_1)+(\rho_2-x)\sigma(\rho_2)}{\rho_2-\rho_1}-\frac{1}{\rho_2-\rho_1}\int\limits_{\rho_1}^{\rho_2}\sigma(x)dx\right|\\ \leq\ &\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\frac{(x-\rho_1)^2\left[|\sigma'(x)|^q+|\sigma'(\rho_1)|^q\right]^{\frac{1}{q}}+(\rho_2-x)^2\left[|\sigma'(x)|^q+|\sigma'(\rho_2)|^q\right]^{\frac{1}{q}}}{\rho_2-\rho_1}\right].\end{aligned}$$

holds for any $x\in[\rho_1,\rho_2]$ if $\sigma'\in L[\rho_1,\rho_2]$ and $|\sigma'|^q$ is convex.

Theorem 1.6 ([11]). *Let $\rho_1,\rho_2 \in I^\circ$(interior of I) with $\rho_1<\rho_2$, $q>1$ and $\sigma : I^\circ \to \mathbb{R}$ be a differentiable function on I°. Then the inequality*

$$\begin{aligned}&\left|\frac{(x-\rho_1)\sigma(\rho_1)+(\rho_2-x)\sigma(\rho_2)}{\rho_2-\rho_1}-\frac{1}{\rho_2-\rho_1}\int\limits_{\rho_1}^{\rho_2}\sigma(x)dx\right|\\ &\leq\ \frac{1}{2}\left[\frac{(x-\rho_1)^2\left|\sigma'\left(\frac{x+2\rho_1}{3}\right)\right|+(\rho_2-x)^2\left|\sigma'\left(\frac{x+2\rho_2}{3}\right)\right|}{\rho_2-\rho_1}\right].\end{aligned}$$

holds for any $x\in[\rho_1,\rho_2]$ if $\sigma'\in L[\rho_1,\rho_2]$ and $|\sigma'|^q$ is concave on $[\rho_1,\rho_2]$.

The main purpose of the article is to present the parametrized inequality of fractional Hermite-Hadamard type.
For simplicity we denote the function

$$\Delta=\frac{(x-\rho_1)^\zeta((1-\gamma)\sigma(x)+\gamma\sigma(\rho_1))+(\rho_2-x)^\zeta((1-\gamma)\sigma(x)+\gamma\sigma(\rho_2))}{\rho_2-\rho_1}$$

2. Main Results

Lemma 2.1. *Let $\gamma\in\mathbb{R}$, $\rho_1,\rho_2 \in I^\circ$(interior of I) with $\rho_1<\rho_2$, $\zeta>0$ and $\sigma : I^\circ \to \mathbb{R}$ be a differentiable function. If $\sigma'\in L[\rho_1,\rho_2]$, then the following equality*

$$\begin{aligned}\Delta-\frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J^\zeta_{x^-}\sigma(\rho_1)+J^\zeta_{x^+}\sigma(\rho_2)\right]=\ &\frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\int\limits_0^1(t^\zeta-\gamma)\sigma'(tx+(1-t)\rho_1)dt\\ &+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\int\limits_0^1(\gamma-t^\zeta)\sigma'(tx+(1-t)\rho_2)dt \quad (4)\end{aligned}$$

holds for any $x\in[\rho_1,\rho_2]$.

2.1. abd.

Proof. It suffices to note that

$$
\begin{aligned}
I_1 &= \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1 (t^\zeta-\gamma)\sigma'(tx+(1-t)\rho_1)dt \\
&= \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\left[\frac{(t^\zeta-\gamma)\sigma(tx+(1-t)\rho_1)}{x-\rho_1}\Big|_0^1 - \frac{\zeta}{x-\rho_1}\int_0^1 t^{\zeta-1}\sigma(tx+(1-t)\rho_1)dt\right] \\
&= \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\left[\frac{(1-\gamma)\sigma(x)+\gamma\sigma(\rho_1)}{x-\rho_1} - \frac{\zeta}{x-\rho_1}\int_0^1 t^{\zeta-1}\sigma(tx+(1-t)\rho_1)dt\right]. \qquad (5)
\end{aligned}
$$

Let $u = tx+(1-t)\rho_1$ in (5) we have

$$
\begin{aligned}
I_1 &= \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\left[\frac{(1-\gamma)\sigma(x)+\gamma\sigma(\rho_1)}{x-\rho_1} - \frac{\zeta}{x-\rho_1}\int_{\rho_1}^{x}\left(\frac{u-\rho_1}{x-\rho_1}\right)^{\zeta-1}\sigma(u)\frac{du}{x-\rho_1}\right] \\
&= \frac{(x-\rho_1)^{\zeta}((1-\gamma)\sigma(x)+\gamma\sigma(\rho_1))}{\rho_2-\rho_1} - \frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}J^{\zeta}_{x^-}\sigma(\rho_1), \qquad (6)
\end{aligned}
$$

similarly

$$
\begin{aligned}
I_2 &= \frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1 (\gamma-t^\zeta)\sigma'(tx+(1-t)\rho_2)dt \\
&= \frac{(\rho_2-x)^{\zeta}((1-\gamma)\sigma(x)+\gamma\sigma(\rho_2))}{\rho_2-\rho_1} - \frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}J^{\zeta}_{x^+}\sigma(\rho_2), \qquad (7)
\end{aligned}
$$

now by adding (6) and (7) we get (4). □

Lemma 2.2. *Let γ be a real number and $\zeta > 0$. Then*

$$
\int_0^1 |\gamma - t^\zeta|dt = \begin{cases} \frac{(\zeta+1)\gamma-1}{\zeta+1} & \gamma \geq 1 \\ \frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}} - 2\gamma^{\frac{\zeta+1}{\zeta}} - \gamma(\zeta+1)+1}{\zeta+1} & 0<\gamma<1 \\ \frac{1-(\zeta+1)\gamma}{\zeta+1} & \gamma \leq 0. \end{cases}
$$

Proof. **Case 1:** If $\gamma \geq 1$, then

$$
\int_0^1 |\gamma - t^\zeta|dt = \int_0^1 (\gamma - t^\zeta)dt = \frac{(\zeta+1)\gamma-1}{\zeta+1}.
$$

Case 2: If $0<\gamma<1$, then

$$
\int_0^1 |\gamma - t^\zeta|dt = \int_0^{\gamma^{\frac{1}{\zeta}}} (\gamma - t^\zeta)dt + \int_{\gamma^{\frac{1}{\zeta}}}^1 (t^\zeta - \gamma)dt = \frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}} - 2\gamma^{\frac{\zeta+1}{\zeta}} - \gamma(\zeta+1)+1}{\zeta+1}.
$$

Case 3: If $\gamma \leq 0$, then

$$
\int_0^1 |\gamma - t^\zeta|dt = \int_0^1 (t-\gamma)dt = \frac{1-(\zeta+1)\gamma}{\zeta+1}.
$$

□

Lemma 2.3. *Let γ be a real number and $\zeta > 0$. Then*

$$\int_0^1 |t^{\zeta}-\gamma| t\,dt = \begin{cases} \frac{(\zeta+2)\gamma-2}{2(\zeta+2)} & \gamma \geq 1 \\ \frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)} & 0<\gamma<1 \\ \frac{2-(\zeta+2)\gamma}{2(\zeta+2)} & \gamma \leq 0. \end{cases}$$

Lemma 2.4. *Let γ be a real number and $\zeta > 0$. Then*

$$\int_0^1 |t^{\zeta}-\gamma|(1-t)\,dt = \begin{cases} \frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)} & \gamma \geq 1 \\ \frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)} & 0<\gamma<1 \\ \frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)} & \gamma \leq 0. \end{cases}$$

Theorem 2.5. *Let $\gamma \in \mathbb{R}$, $\rho_1, \rho_2 \in I^{\circ}$(interior of I) with $\rho_1 < \rho_2$, $\zeta > 0$ and $\sigma : I^{\circ} \to \mathbb{R}$ be a differentiable function. Then the following inequality*

$$\begin{aligned} &\left| \Delta - \frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J^{\zeta}_{x^-}\sigma(\rho_1)+J^{\zeta}_{x^+}\sigma(\rho_2)\right] \right| \\ \leq\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1} \begin{cases} \left\{ |\sigma'(x)|\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right) \right\} \text{ if } \gamma \geq 1 \\ \left\{ \begin{aligned} &|\sigma'(x)|\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right) \\ &+|\sigma'(\rho_1)|\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right) \end{aligned} \right\} \text{ if } 0<\gamma<1 \\ \left\{ |\sigma'(x)|\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right) \right\} \text{ if } \gamma \leq 0 \end{cases} \\ &+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1} \begin{cases} \left\{ |\sigma'(x)|\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right) \right\} \text{ if } \gamma \geq 1 \\ \left\{ \begin{aligned} &|\sigma'(x)|\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right) \\ &+|\sigma'(\rho_2)|\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right) \end{aligned} \right\} \text{ if } 0<\gamma<1 \\ \left\{ |\sigma'(x)|\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right) \right\} \text{ if } \gamma \leq 0. \end{cases} \end{aligned}$$

holds for any $x \in [\rho_1, \rho_2]$ if $\sigma' \in L[\rho_1, \rho_2]$ and $|\sigma'|$ is convex.

Proof. Using Lemma 2.1 it follows that

$$\begin{aligned}
&\left|\Delta-\frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J_{x^-}^{\zeta}\sigma(\rho_1)+J_{x^+}^{\zeta}\sigma(\rho_2)\right]\right| \\
\leq\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1|t^{\zeta}-\gamma||\sigma'(tx+(1-t)\rho_1)|dt+\frac{(\rho_2-x)^2}{\rho_2-\rho_1}\int_0^1|\gamma-t^{\zeta}||\sigma'(tx+(1-t)\rho_2)|dt \\
\leq\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1|t^{\zeta}-\gamma|[t|\sigma'(x)|+(1-t)|\sigma'(\rho_1)|]dt \\
&+\frac{(\rho_2-x)^2}{\rho_2-\rho_1}\int_0^1|\gamma-t^{\zeta}|[t|\sigma'(x)|+(1-t)|\sigma'(\rho_2)|]dt
\end{aligned}$$

$$= \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\begin{cases}\left\{|\sigma'(x)|\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right)\right\} \text{ if } \gamma\geq 1\\ \left\{|\sigma'(x)|\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right)\right. \\ \left.+|\sigma'(\rho_1)|\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right)\right\} \text{ if } 0<\gamma<1\\ \left\{|\sigma'(x)|\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right)\right\} \text{ if } \gamma\leq 0\end{cases}$$

$$+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\begin{cases}\left\{|\sigma'(x)|\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right)\right\} \text{ if } \gamma\geq 1\\ \left\{|\sigma'(x)|\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right)\right. \\ \left.+|\sigma'(\rho_2)|\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right)\right\} \text{ if } 0<\gamma<1\\ \left\{|\sigma'(x)|\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right)\right\} \text{ if } \gamma\leq 0.\end{cases}$$

□

Remark 2.6. *If we select $\zeta=1$, in Theorem 2.5 we obtain Theorem 2.1 in [12].*

Theorem 2.7. *Let $\gamma\in\mathbb{R}$, $\zeta>0$, $\rho_1,\rho_2\in I^{\circ}$(interior of I) with $\rho_1<\rho_2$, $q>1$ and $\sigma:I^{\circ}\to\mathbb{R}$ be a differentiable function on I°. Then the inequality*

$$
\begin{aligned}
&\left|\Delta-\frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J_{x^-}^{\zeta}\sigma(\rho_1)+J_{x^+}^{\zeta}\sigma(\rho_2)\right]\right| \\
\leq\quad & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}
\begin{cases}
\left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|^q\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\geq 1\\
\left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right)\right. & \\
\left.+|\sigma'(\rho_1)|^q\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } 0<\gamma<1\\
\left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|^q\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\leq 0
\end{cases}\\
&+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}
\begin{cases}
\left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|^q\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\geq 1\\
\left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right)\right. & \\
\left.+|\sigma'(\rho_2)|^q\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } 0<\gamma<1\\
\left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|^q\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\leq 0
\end{cases}
\end{aligned}
$$

holds for any $x\in[\rho_1,\rho_2]$ if $\sigma'\in L[\rho_1,\rho_2]$ and $|\sigma'|^q$ is convex.

Proof. Using Lemma 2.1 and Power mean inequality, we have

$$\begin{aligned}
&\left|\Delta-\frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J^{\zeta}_{x^-}\sigma(\rho_1)+J^{\zeta}_{x^+}\sigma(\rho_2)\right]\right|\\
\leq\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1|t^{\zeta}-\gamma||\sigma'(tx+(1-t)\rho_1)|dt+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1|\gamma-t^{\zeta}||\sigma'(tx+(1-t)\rho_2)|dt\\
\leq\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\left(\int_0^1|t^{\zeta}-\gamma|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(|t^{\zeta}-\gamma|)|\sigma'(tx+(1-t)\rho_1)|^q dt\right)^{\frac{1}{q}}\\
+\ & \frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\left(\int_0^1|\gamma-t^{\zeta}|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\gamma-t^{\zeta}||\sigma'(tx+(1-t)\rho_2)|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\left(\int_0^1|t^{\zeta}-\gamma|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t^{\zeta}-\gamma|[t|\sigma'(x)|^q+(1-t)|\sigma'(\rho_1)|^q]dt\right)^{\frac{1}{q}}\\
+\ & \frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\left(\int_0^1|\gamma-t^{\zeta}|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\gamma-t^{\zeta}|[t|\sigma'(x)|^q+(1-t)|\sigma'(\rho_2)|^q]dt\right)^{\frac{1}{q}}\\
=\ & \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\begin{cases}
\left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|^q\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\geq 1\\
\left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right)\right. & \\
\left.+|\sigma'(\rho_1)|^q\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } 0<\gamma<1\\
\left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_1)|^q\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\leq 0
\end{cases}\\
&+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\begin{cases}
\left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|f'(x)|^q\left(\frac{(\zeta+2)\gamma-2}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|^q\left(\frac{(\zeta+1)(\zeta+2)\gamma-2}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\geq 1\\
\left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-\gamma(\zeta+2)+2}{2(\zeta+2)}\right)\right. & \\
\left.+|\sigma'(\rho_2)|^q\left(\frac{2-\zeta^2\gamma-3\zeta\gamma-2\gamma+8\zeta\gamma^{\frac{\zeta+1}{\zeta}}+4\zeta^2\gamma^{\frac{\zeta+1}{\zeta}}-2\zeta\gamma^{\frac{\zeta+2}{\zeta}}-2\zeta^2\gamma^{\frac{\zeta+2}{\zeta}}}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } 0<\gamma<1\\
\left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)^{1-\frac{1}{q}}\left[|\sigma'(x)|^q\left(\frac{2-(\zeta+2)\gamma}{2(\zeta+2)}\right)+|\sigma'(\rho_2)|^q\left(\frac{2-(\zeta+1)(\zeta+2)\gamma}{2(\zeta+1)(\zeta+2)}\right)\right]^{\frac{1}{q}} & \text{if } \gamma\leq 0.
\end{cases}
\end{aligned}$$

□

Remark 2.8. *If we select $\zeta=1$, in Theorem 2.7 we obtain Theorem 2.2 in [12].*

Theorem 2.9. *Let $\gamma \in \mathbb{R}$, $\zeta > 0$, $\rho_1, \rho_2 \in I^\circ$(interior of I) with $\rho_1 < \rho_2$, $q > 1$ and $\sigma : I^\circ \to \mathbb{R}$ be a differentiable function on I°. Then the inequality*

$$\left| \Delta - \frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J^{\zeta}_{x^-}\sigma(\rho_1) + J^{\zeta}_{x^+}\sigma(\rho_2) \right] \right|$$

$$\leq \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1} \begin{cases} \left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)\left|\sigma'\left(\frac{(2-\gamma(\zeta+2))(\zeta+1)x+(2-2\gamma(\zeta+1)(\zeta+2)+\gamma(\zeta+1)(\zeta+2))\rho_1}{2(\zeta+2)(1-\gamma(\zeta+1))}\right)\right| \\ \text{if } \gamma \geq 1 \\ \left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right) \\ \left|\sigma'\left(\frac{(2-\gamma(\zeta+2))(\zeta+1)x+(2-2\gamma(\zeta+1)(\zeta+2)+\gamma(\zeta+1)(\zeta+2))\rho_1}{2(\zeta+2)(1-\gamma(\zeta+1))}\right)\right| \text{ if } 0<\gamma<1 \\ \left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)\left|\sigma'\left(\frac{(2-\gamma(\zeta+2))(\zeta+1)x+(2-2\gamma(\zeta+1)(\zeta+2)+\gamma(\zeta+1)(\zeta+2))\rho_1}{2(\zeta+2)(1-\gamma(\zeta+1))}\right)\right| \\ \text{if } \gamma \leq 0 \end{cases}$$

$$+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1} \begin{cases} \left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)\left|\sigma'\left(\frac{(\gamma(\zeta+2)-2)(\zeta+1)x+(2\gamma(\zeta+1)(\zeta+2)-\gamma(\zeta+1)(\zeta+2)-2)\rho_2}{2(\zeta+2)(\gamma(\zeta+1)-1)}\right)\right| \\ \text{if } \gamma \geq 1 \\ \left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right) \\ \left|\sigma'\left(\frac{(\gamma(\zeta+2)-2)(\zeta+1)x+(2\gamma(\zeta+1)(\zeta+2)-\gamma(\zeta+1)(\zeta+2)-2)\rho_2}{2(\zeta+2)(\gamma(\zeta+1)-1)}\right)\right| \text{ if } 0<\gamma<1 \\ \left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)\left|\sigma'\left(\frac{(\gamma(\zeta+2)-2)(\zeta+1)x+(2\gamma(\zeta+1)(\zeta+2)-\gamma(\zeta+1)(\zeta+2)-2)\rho_2}{2(\zeta+2)(\gamma(\zeta+1)-1)}\right)\right| \\ \text{if } \gamma \leq 0 \end{cases}$$

holds for any $x \in [\rho_1, \rho_2]$ if $\sigma' \in L[\rho_1, \rho_2]$ and $|\sigma'|$ is concave.

Proof. It follows from [30] and the concavity of $|\sigma'|^q$ that $|\sigma'|$ is also concave. Making use of Lemma 2.1 and Jensen's integral inequality one has

$$\left|\Delta-\frac{\Gamma(\zeta+1)}{\rho_2-\rho_1}\left[J^{\zeta}_{x^-}\sigma(\rho_1)+J^{\zeta}_{x^+}\sigma(\rho_2)\right]\right|$$

$$\leq \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1|t^{\zeta}-\gamma||\sigma'(tx+(1-t)\rho_1)|dt+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\int_0^1|\gamma-t^{\zeta}||\sigma'(tx+(1-t)\rho_2)|dt$$

$$\leq \frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\left(\int_0^1|t^{\zeta}-\gamma|dt\right)\left|\sigma'\left(\frac{\int_0^1|t^{\zeta}-\gamma|(tx+(1-t)\rho_1)dt}{\int_0^1|t^{\zeta}-\gamma|dt}\right)\right|$$

$$+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\left(\int_0^1|\gamma-t^{\zeta}|dt\right)\left|\sigma'\left(\frac{\int_0^1|\gamma-t^{\zeta}|(tx+(1-t)\rho_2)dt}{\int_0^1|\gamma-t^{\zeta}|dt}\right)\right|$$

$$=\frac{(x-\rho_1)^{\zeta+1}}{\rho_2-\rho_1}\begin{cases}\left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)\left|\sigma'\left(\frac{(2-\gamma(\zeta+2))(\zeta+1)x+(2-2\gamma(\zeta+1)(\zeta+2)+\gamma(\zeta+1)(\zeta+2))\rho_1}{2(\zeta+2)(1-\gamma(\zeta+1))}\right)\right| \\ \text{if } \gamma\geq 1 \\ \left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right) \\ \left|\sigma'\left(\frac{(2-\gamma(\zeta+2))(\zeta+1)x+(2-2\gamma(\zeta+1)(\zeta+2)+\gamma(\zeta+1)(\zeta+2))\rho_1}{2(\zeta+2)(1-\gamma(\zeta+1))}\right)\right| \text{ if } 0<\gamma<1 \\ \left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)\left|\sigma'\left(\frac{(2-\gamma(\zeta+2))(\zeta+1)x+(2-2\gamma(\zeta+1)(\zeta+2)+\gamma(\zeta+1)(\zeta+2))\rho_1}{2(\zeta+2)(1-\gamma(\zeta+1))}\right)\right| \\ \text{if } \gamma\leq 0\end{cases}$$

$$+\frac{(\rho_2-x)^{\zeta+1}}{\rho_2-\rho_1}\begin{cases}\left(\frac{(\zeta+1)\gamma-1}{\zeta+1}\right)\left|\sigma'\left(\frac{(\gamma(\zeta+2)-2)(\zeta+1)x+(2\gamma(\zeta+1)(\zeta+2)-\gamma(\zeta+1)(\zeta+2)-2)\rho_2}{2(\zeta+2)(\gamma(\zeta+1)-1)}\right)\right| \\ \text{if } \gamma\geq 1 \\ \left(\frac{2(\zeta+1)\gamma^{\frac{\zeta+1}{\zeta}}-2\gamma^{\frac{\zeta+1}{\zeta}}-\gamma(\zeta+1)+1}{\zeta+1}\right) \\ \left|\sigma'\left(\frac{(\gamma(\zeta+2)-2)(\zeta+1)x+(2\gamma(\zeta+1)(\zeta+2)-\gamma(\zeta+1)(\zeta+2)-2)\rho_2}{2(\zeta+2)(\gamma(\zeta+1)-1)}\right)\right| \text{ if } 0<\gamma<1 \\ \left(\frac{1-(\zeta+1)\gamma}{\zeta+1}\right)\left|\sigma'\left(\frac{(\gamma(\zeta+2)-2)(\zeta+1)x+(2\gamma(\zeta+1)(\zeta+2)-\gamma(\zeta+1)(\zeta+2)-2)\rho_2}{2(\zeta+2)(\gamma(\zeta+1)-1)}\right)\right| \\ \text{if } \gamma\leq 0.\end{cases}$$

□

Remark 2.10. *If we select $\zeta=1$, in Theorem 2.9 we obtain Theorem 2.3 in [12].*

3. Conclusion

In this work, a parameterized inequality of fractional Hermite-Hadamard has been presented.

Competing interests

The authors declare that they have no competing interests.

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Author's contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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