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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR $s-$CONVEX FUNCTIONS OF MIXED KIND

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Abstract. The s-convex functions of first kind and of second kind are well known functions. We would use recently introduced notion of $s-$convex functions of mixed kind and we would call it $(s,r)-$convex function. It would generalize the notion of first kind and second kind convexity in the sense that both kinds of convexities would be obtained easily by imposing certain specific conditions on it. We would state generalization of Hermite-Hadamard type inequalities by using our new notion of $(s,r)-$convex function via Riemann-Lioville fractional integrals.

Keywords: Convex functions, $s-$convex functions, Hermite-Hadamard inequalities.

1. INTRODUCTION AND PRELIMINARY RESULTS

The theory of convex functions is getting attention of people significantly nowadays. It is observed that by using different and new ideas and different ways classical convexity has become general and extended see for example [2, 4, 6, 10, 16–18, 20]. The notion of $s-$convex function as $s-$convex function of 2nd kind in literature was introduced by Breckner [2] as follows:

Definition 1.1. *Let $I \subset [0,\infty)$. A function $\Psi : I \to \mathbb{R}$ is said to be $s-$convex of the 2nd kind, if*

$$\Psi((1-w)x+wz) \leq (1-w)^s\Psi(x)+w^s\Psi(y) \tag{1}$$

for all $x,z \in I$; $s,w \in [0,1]$.

For definition of first kind convex functions we recall from [11]:

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Definition 1.2. *Let I be a real interval. A function $\Psi : I \to \mathbb{R}$ is known as $s-$convex of the $1st$ kind, if*

$$\Psi((1-w)x+wz) \leq (1-w^r)\Psi(x)+w^r\Psi(z) \tag{2}$$

for all $x,z \in I$; $r,w \in [0,1]$.

It is worth mentioning that in article [11], the definition of first kind is cited to Toaders' article [17], but we could not find such definition in [17]. So we consider it as a new definition of first kind in sense of Noor and Awan article [11].

In [8] we can find definition of $(s,m)-$convex function given by Mihesan in year 1993 which may be stated as:

Definition 1.3. *The function $f : [0,b] \to \mathbb{R}$ with $b > 0$ is said to be $(s,m)-$convex, where $s,m \in (0,1]$, and $w \in [0,1]$ if*

$$f(tx+m(1-w)y) \leq t^s f(x)+m(1-w^s)f(y) \tag{3}$$

Note that if $m = 1$, then $(s,m)-$convexity make match with definition of first kind as stated in 1.2.

One of the well-known inequalities related to the integral mean of a convex functions are the Hermite-Hadamard Inequalities named after Charles Hermite and Jacques Hadamard and sometimes called Hadamard's inequality [7]. The Hermite-Hadamard inequality is a refinement of midpoint inequality which is a special case of Jensen's inequality. Due to the importance of Hermite-Hadamard inequality, a number of mathematicians continue to write research articles on this type of inequalities. The Hermite-Hadamard inequality has made great contributions in the fields of integral inequalities, approximation theory, information theory, special means theory and numerical analysis and has been developed for different classes of convexities. Hermite-Hadamard type of inequalities ware established by Sarikaya $et.\ al.$ [14] through fractional integrals one can similar inequalities in articles [10, 13, 19].

In this paper we would recall some results for 1st and 2nd kind of $s-$convex functions. We would like to use newly introduce class of $s-$convex function of mixed kind. We would apply these results to Hermite-Hadamard type inequalities via fractional integral. The idea used in this work may attract research in the near future. This is the main motivation of the paper. From now onward we assume that, unless stated otherwise, I is a real interval.

We recall here definition of $P-$convex function from [6]:

Definition 1.4. *A function $\psi : I \to \mathbb{R}$ is known as $P-$convex, if*

$$\psi((1-w)x+wz) \leq \psi(x)+\psi(z) \tag{4}$$

for all $x,z \in I$ and $w \in [0,1]$. This class of functions is denoted by $P(I)$.

Here we also have definition of quasi$-$convex function (for detailed discussion see [3] and [5])

Definition 1.5. *A function $\psi : I \to \mathbb{R}$ is known as quasi$-$convex, if*

$$\psi((1-w)x+wz) \leq \max\{\psi(x), \psi(z)\} \tag{5}$$

for all $x,z \in I$, $w \in [0,1]$.

Now we slightly modify the definition of $s-$convex functions of the first kind as follows:

Definition 1.6. *A function $\psi : I \to \mathbb{R}$ will be called $s-$convex of the 1^{st} kind, if*

$$\psi((1-w)x+wz) \leq (1-w^r)\psi(x)+w^r\psi(z) \tag{6}$$

for all $x,z \in I$; $r,w \in [0,1]$.

Remark 1.7. *Note that in this definition we also included $r = 0$. If we put $r = 0$, we easily get quasi$-$convexity provided that $\psi(z) \geq \psi(x)$.*

Again we modify here definition of second kind convexity as under:

Definition 1.8. *A function $\psi : I \subset [0,\infty) \to \mathbb{R}$, will be called $s-$convex of the 2^{nd} kind, if*

$$\psi((1-w)x+wz) \leq (1-w)^s\psi(x)+w^s\psi(z) \tag{7}$$

for all $x,z \in I$; $s,w \in [0,1]$.

Remark 1.9. *In the similar manner, we have slightly improved definition of second kind convexity by including $s = 0$. If we put $s = 0$, we easily get $P-$convexity (see Definition 1.4).*

In [1] we have recently introduced a new class of functions, $s-$convex functions in the mixed kind, for simplicity we would call it class of $(s,r)-$convex functions, may be defined as:

Definition 1.10. *A function $\psi : I \to \mathbb{R}$ is $(s,r)-$convex if*

$$\psi((1-w)x+wz) \leq (1-w^r)^s\psi(x)+w^{rs}\psi(z), \tag{8}$$

for all $x,z \in I$; $r,s,w \in [0,1]$.

Remark 1.11.
For $s = 1$, (8) gives us 1^{st} kind convexity provided that $r \neq 0$.
For $r = 1$, (8) gives us 2^{nd} kind convexity provided that $s \neq 0$.
If s and r both are equal to 1, then (8) gives us ordinary convexity.
If $s \in (0,1)$ is arbitrary and $r = 0$, then (8) gives us quasi$-$convexity provided that $\psi(z) \geq \psi(x)$.
If $s = 0$ and r is arbitrary, then (8) gives us $P-$convexity provided that $r \neq 0$.

Renowned Hölder's inequality in its general integral form is given as follows [9]:

Theorem 1.12. *Let $1 \leq p,q \leq \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. If $\psi \in L_p$ and $\phi \in L_q$, then $\psi\phi \in L_1$ and*

$$\int |\psi(u)\phi(u)|du \leq \|\psi\|_p\|\phi\|_q \tag{9}$$

where $\psi \in L_p$ if $\|\psi\|_p = \left(\int |\psi(u)|^p du\right)^{\frac{1}{p}} < \infty$.

Note that if we put $p = q = 2$, the above inequality becomes Cauchy – Schwarz inequality. Also, if we put $q = 1$ and let $p \to \infty$, then we get,

$$\int |\psi(u)\phi(u)|du \leq ||\psi||_\infty||\phi||_1$$

where $||\psi||_\infty$ stands for the essential supremum of $|\psi|$, i.e.,

$$||\psi||_\infty = ess\sup_{\forall u}|\psi(u)|.$$

Definition 1.13. *Let ψ, ϕ are real valued functions defined on $[z_1,z_2]$ and if $|\psi|$ and $|\psi||\phi|^q$ are integrable on $[z_1,z_2]$, then for $q \geq 1$ we have:*

$$\int_{z_1}^{z_2} |\psi(u)||\phi(u)|du \leq \left(\int_{z_1}^{z_2} |\psi(u)|du\right)^{1-\frac{1}{q}} \left(\int_{z_1}^{z_2} |\psi(u)||\phi(u)|^q du\right)^{\frac{1}{q}}.$$

The above inequality is known as power mean inequality (see for reference [12]).

Definition 1.14. *[15] Let $\psi \in L[z_1,z_2]$. Then the Reimann−Liouville fractional integrals $J^{\eta}_{z_1^+}\psi(\theta)$ and $J^{\eta}_{z_2^-}\psi(\theta)$ of order $\eta > 0$ and $z_1 \geq 0$ are defined as respectively:*

$$J^{\eta}_{z_1^+}\psi(\theta) = \frac{1}{\Gamma(\eta)}\int_{z_1}^{\theta}(\theta-t)^{\eta-1}\psi(t)dt, \qquad \theta > z_1$$

and

$$J^{\eta}_{z_2^-}\psi(\theta) = \frac{1}{\Gamma(\eta)}\int_{\theta}^{z_2}(t-\theta)^{\eta-1}\psi(t)dt, \qquad \theta < z_2,$$

where

$$\Gamma(\eta) = \int_0^{\infty} e^{-t}t^{\eta-1}dt.$$

Moreover, $J^0_{z_1^+}\psi(\theta) = J^0_{z_2^-}\psi(\theta) = \psi(\theta)$. If we put $\eta = 1$, the fractional integral reduces to classical integral.

2. Some Significant Results

All the following significant results are extracted from [11].

Theorem 2.1. *Let a function $\Psi : [c,d] \to \mathbb{R}$ be two times differentiable function on (c,d) with $c < d$. Suppose $\Psi'' \in L[c,d]$ and $|\Psi''|$ is s-convex function of 2nd kind, then the inequality for the integral fraction is given as follows;*

$$\left|\frac{2^{\eta-1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c) + J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right] - \Psi\left(\frac{c+d}{2}\right)\right| \leq \frac{(d-c)^2}{2^{s+3}(\eta+1)}\left[C(s,\eta,w) + \frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right]\left[|\Psi''(c)| + |\Psi''(d|\right].$$

where $C(s,\eta,w) = \int_0^1 (1-w)^{(\eta+1)}(1+w)^s dw$.

Theorem 2.2. *Let a function $\Psi : [c,d] \to \mathbb{R}$ be two times differentiable function on (c,d) with $c < d$. Let $\Psi'' \in L[c,d]$ and $|\Psi''|$ is s-convex function of 2nd kind, then integral fraction inequality is given as follows:*

$$\left|\frac{2^{\eta-1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c) + J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right] - \Psi\left(\frac{c+d}{2}\right)\right|$$
$$\leq \frac{(d-c)^2}{2^{\frac{3q+s}{q}}}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\left[|\Psi''(c)|^q(2^{s+1}-1) + |\Psi''(d)|^q\right]^{\frac{1}{q}} + \left[|\Psi''(d)|^q(2^{s+1}-1) + |\Psi''(c)|^q\right]^{\frac{1}{q}}\right\},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $q\geq 1$.

Theorem 2.3. *Let a function* $\Psi:[c,d]\to\mathbb{R}$ *be two times differentiable function on* (c,d) *with* $c<d$. *If* $\Psi''\in L[c,d]$ *and* $|\Psi''|^q$ *for* $q\geq 1$ *is s-convex function of* 2*nd kind, then integral fraction inequality is given as follows;*

$$\left|\frac{2^{\eta-1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$
$$\leq\frac{(d-c)^2}{2^{\frac{3q+s}{q}}(\eta+1)(\eta+2)^{1-\frac{1}{q}}}\left\{\left(|\Psi''(c)|^q(C(s,\eta,w))+|\Psi''(d)|^q\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\Psi''(d)|^q(C(s,\eta,w))+|\Psi''(c)|^q\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right)^{\frac{1}{q}}\right\},$$

where $C(s,\eta,w)=\displaystyle\int_0^1(1-w)^{(\eta+1)}(1+w)^s dw$.

Theorem 2.4. *Let a function* $\Psi:[c,d]\to\mathbb{R}$ *be two times differentiable function on* (c,d) *with* $c<d$. *Let* $\Psi''\in L[c,d]$ *and* $|\Psi''|$ *is s-convex function of* 1*st kind, then integral fraction inequality is given as follows;*

$$\left|\frac{2^{\eta-1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|\leq\frac{(d-c)^2}{8(\eta+1)(\eta+2)}\left[|\Psi''(c)|+|\Psi''(d)|\right].$$

Theorem 2.5. *Let a function* $\Psi:[c,d]\to\mathbb{R}$ *be two times differentiable function on* (c,d) *with* $c<d$. *If* $\Psi''\in L[c,d]$ *and* $|\Psi''|$ *is s-convex function of* 1*st kind, then integral fraction inequality is given as follows;*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$
$$\leq\frac{(d-c)^2}{2^{\frac{s+3q}{q}}(\eta+1)}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\times$$
$$\left\{\left(|\Psi''(c)|^q(2^r(r+1)-1)+|\Psi''(d)|^q\right)^{\frac{1}{q}}+\left(|\Psi''(d)|^q(2^s(s+1)-1)+|\Psi''(c)|^q\right)^{\frac{1}{q}}\right\},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $q\geq 1$.

Theorem 2.6. *Let a function* $\Psi:[c,d]\to\mathbb{R}$ *be two times differentiable function on* (c,d) *with* $c<d$. *Let* $\Psi''\in L[c,d]$ *and* $|\Psi''|^q$ *for* $q\geq 1$ *is s-convex function of* 1*st kind, then integral fraction inequality is given as follows;*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|\leq\frac{(d-c)^2}{8(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times$$
$$\left\{\left[|\Psi''(c)|^q\left(\frac{1}{\eta+1}-\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right)+\Psi''(d)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right]^{\frac{1}{q}}+\right.$$
$$\left.\left[|\Psi''(d)|^q\left(\frac{1}{\eta+1}-\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right)+\Psi''(c)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right]^{\frac{1}{q}}\right\}$$

Now in the next section we will state our main results.

3. Main Results

Before proceeding further, we state here an important result from [11], that will help us to solve our main results:

Lemma 3.1. *Let a function $\Psi:[c,d]\to\mathbb{R}$ be two times differentiable function on (c,d) with $c<d$. If Ψ'' in $L[c,d]$, we have the identity given below for integral fraction:*

$$\frac{2^{\eta-1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^{-}}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^{+}}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)=\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\times$$
$$\left[\Psi''\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)+\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right]dw.$$

In this section we will obtain three main results for Hermite-Hadamard types fractional integral of (s,r)-convex function of mixed kind.

3.1. Our first main results for (s,r)-convex function.

Theorem 3.2. *Let a function $\Psi:[c,d]\to\mathbb{R}$ be two times differentiable function on (c,d) with $c<d$. Let $\Psi''\in L[c,d]$ and $|\Psi''|$ is the (s,r)-convex function, then integral fraction inequality is given as follows;*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^{-}}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^{+}}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$
$$\leq\frac{(d-c)^2}{2^{rs+3}(\eta+1)}\left\{H(r,s,\eta,w)+\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}\right\}\left[|\Psi''(c)|+|\Psi''(d)|\right]. \tag{10}$$

where $H(r,s,\eta,w)=\int_0^1(1-w)^{(\eta+1)}[2^r-(1-w)^r]^s dw.$

Proof. By using Lemma 3.1, property of absolute value function and the fact that $|\Psi''|$ is $(s,r)-$convex function, we have

$$
\begin{aligned}
&\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right| \\
&=\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\Psi''\left[\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)+\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right]dw\right| \\
&\leq\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\left[\Psi''\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)\right]dw\right| \\
&+\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\left[\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right]dw\right| \\
&\leq\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\left|\Psi''\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)\right|dw \\
&+\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\left|\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right|dw \\
&\leq\frac{(d-c)^2}{8(\eta+1)}\left\{\int_0^1(1-w)^{(\eta+1)}\left|\Psi''\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)\right|dw\right. \\
&\left.+\int_0^1(1-w)^{(\eta+1)}\left|\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right|dw\right\} \\
&\leq\frac{(d-c)^2}{8(\eta+1)}\left\{\int_0^1(1-w)^{(\eta+1)}\left(\left[1-\left(\frac{1-w}{2}\right)^r\right]^s|\Psi''(c)|+\left(\frac{1-w}{2}\right)^{rs}|\Psi''(d)|\right)dw\right. \\
&\left.+\int_0^1(1-w)^{(\eta+1)}\left(\left[1-\left(\frac{1-w}{2}\right)^r\right]^s|\Psi''(d)|+\left(\frac{1-w}{2}\right)^{rs}|\Psi''(c)|\right)dw\right\} \\
&\leq\frac{(d-c)^2}{2^{rs+3}(\eta+1)}\left\{H(r,s,\eta,w)|\Psi''(c)|+\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}|\Psi''(d)|\right. \\
&\left.+H(r,s,\eta,w)|\Psi''(d)|+\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}|\Psi''(c)|\right\} \\
&\leq\frac{(d-c)^2}{2^{rs+3}(\eta+1)}\left\{H(r,s,\eta,w)+\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}\right\}\left[|\Psi''(c)|+|\Psi''(d)|\right].
\end{aligned}
$$

□

Corollary 3.3. *If we put $r=1$ in Theorem 3.2, then (10) becomes*

$$
\begin{aligned}
&\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right| \\
&\qquad\leq\frac{(d-c)^2}{2^{3+s}(\eta+1)}\left\{H(1,s,\eta,w)+\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right\}\left[|\Psi''(c)|+|\Psi''(d)|\right]
\end{aligned}
$$

which is equal to the Theorem 2.1 and Theorem 2 in [11].

Corollary 3.4. *If we put $s=1$ in Theorem 3.2, then (10) becomes*

$$
\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})}-\Psi(c)+J^{\eta}_{(\frac{c+d}{2})}+\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|\leq\frac{(d-c)^2}{8(\eta+1)(\eta+2)}\left[|\Psi''(c)|+|\Psi''(d)|\right]
$$

which is equal to the Theorem 2.4 and Theorem 5 in [11].

3.2. **Our second main results for (s,r)-convex function.**

Theorem 3.5. *Let a function $\Psi:[c,d]\to\mathbb{R}$ be two times differentiable function on (c,d) with $c<d$.Let $\Psi''\in L[c,d]$ and $|\Psi''|$ is the $(s,r)-$convex function for mixed kind.*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|\le\frac{(d-c)^2}{2^{\frac{rs+3q}{q}}(\eta+1)}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\times$$
$$\left\{\left(|\Psi''(c)|^qG(r,s,\eta,w)+\frac{|\Psi''(d)|^q}{(rs+1)}\right)^{\frac{1}{q}}+\left(|\Psi''(d)|^qG(r,s,\eta,w)+\frac{|\Psi''(c)|^q}{(rs+1)}\right)^{\frac{1}{q}}\right\}. \tag{11}$$

where $G(r,s,\eta,w)=\int_0^1[2^r-(1-w)^r]^s dw$, $\frac{1}{p}+\frac{1}{q}=1$ *with* $q\ge 1$.

Proof. By using Lemma 3.1, Hölder's inequality and the fact that $|\Psi''|$ is $(s,r)-$convex function, we have

$$\begin{aligned}
&\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|\\
&=\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\Psi''\left[\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)+\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right]dw\right|\\
&\le\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\Psi''\left[\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)dw\right]\right|\\
&+\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\Psi''\left[\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)dw\right]\right|\\
&\le\frac{(d-c)^2}{8(\eta+1)}\left\{\int_0^1(1-w)^{(\eta+1)}\Psi''\left[\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)\right]dw\right.\\
&\left.+\int_0^1(1-w)^{(\eta+1)}\left[\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right]dw\right\}\\
&\le\frac{(d-c)^2}{8(\eta+1)}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\times\\
&\left\{\left[|\Psi''(c)|^q\int_0^1\left[1-\left(\frac{1-w}{2}\right)^r\right]^s dw+\left(|\Psi''(d)|^q\int_0^1(\frac{1-w}{2})^{rs}dw\right)\right]^{\frac{1}{q}}\right.\\
&\left.+\left(|\Psi''(d)|^q\int_0^1\left[1-\left(\frac{1-w}{2}\right)^r\right]^s dw+\left(|\Psi''(c)|^q\int_0^1(\frac{1-w}{2})^{rs}dw\right)\right)^{\frac{1}{q}}\right\}\\
&\le\frac{(d-c)^2}{2^{\frac{rs+3q}{q}}(\eta+1)}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\left\{\left(|\Psi''(c)|^qG(r,s,\eta,w)+\frac{|\Psi''(d)|^q}{(rs+1)}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|\Psi''(d)|^qG(r,s,\eta,w)+\frac{|\Psi''(c)|^q}{(rs+1)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

□

Corollary 3.6. *If $r=1$ in Theorem 3.5, then (11) becomes*

$$\left|\frac{2^{\eta-1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$

$$\leq \frac{(d-c)^2}{2^{\frac{3q+s}{q}}}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\left[|\Psi''(c)|^q(2^{s+1}-1)+|\Psi''(d)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[|\Psi''(d)|^q(2^{s+1}-1)+|\Psi''(c)|^q\right]^{\frac{1}{q}}\right\},$$

which is equal to the Theorem 2.2 and Theorem 3 in [11].

Corollary 3.7. *If $s=1$ in Theorem 3.5, then (11) becomes*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$

$$\leq \frac{(d-c)^2}{2^{\frac{s+3q}{q}}(\eta+1)}\left(\frac{1}{p(\eta+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\times$$
$$\left\{\left(|\Psi''(c)|^q(2^s(s+1)-1)+|\Psi''(d)|^q\right)^{\frac{1}{q}}+\left(|\Psi''(d)|^q(2^s(s+1)-1)+|\Psi''(c)|^q\right)^{\frac{1}{q}}\right\},$$

which is equal to the Theorem 2.5 and Theorem 6 in [11].

3.3. Our third main results for $(s,r)-$convex function.

Theorem 3.8. *Let a function $\Psi:[c,d]\to\mathbb{R}$ be two times differentiable function on (c,d) with $c<d$.Let $\Psi''\in L[c,d]$ and $|\Psi''|^q$ for $q\geq 1$ is the (s,r)-convex function, then integral fraction inequality is given as follows;*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|\leq\frac{(d-c)^2}{2^{\frac{3q+rs}{q}}(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times$$
$$\left\{\left(H(s,r,\eta,w)|\Psi''(c)|^q+\left(\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}\right)|\Psi''(d)|\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(H(s,r,\eta,w)|\Psi''(d)|^q+\left(\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}\right)|\Psi''(c)|\right)^{\frac{1}{q}}\right\}, \tag{12}$$

where $H(s,r,\eta,w)=\int_0^1(1-w)^{(\eta+1)}(2^r-(1-w)^r)^s$.

Proof. By using Lemma 3.1, the power mean inequality and the fact that $|\Psi''|^q$ is $(s,r)-$convex function, we have

$$
\begin{aligned}
&\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|=\\
&\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\left[\Psi''\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)+\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right]dw\right|\\
&\leq\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\Psi''\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)dw\right|\\
&+\left|\frac{(d-c)^2}{8(\eta+1)}\int_0^1(1-w)^{(\eta+1)}\Psi''\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)dw\right|\\
&\leq\frac{(d-c)^2}{8(\eta+1)}\left\{\left(\int_0^1(1-w)^{(\eta+1)}dw\right)^{1-\frac{1}{q}}\left(\int_0^1(1-w)^{(\eta+1)}\Psi''\left|\left(\frac{1+w}{2}c+\frac{1-w}{2}d\right)\right|\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1(1-w)^{(\eta+1)}dw\right)^{1-\frac{1}{q}}\left(\int_0^1(1-w)^{(\eta+1)}\Psi''\left|\left(\frac{1+w}{2}d+\frac{1-w}{2}c\right)\right|\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{(d-c)^2}{8(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times\\
&\left\{\left(\int_0^1(1-w)^{(\eta+1)}\left[\left(1-\frac{1-w}{2}\right)^r\right]^s|\Psi''(c)|^q+\left(\frac{1-w}{2}\right)^{rs}|\Psi''(d)|^q dw\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1(1-w)^{(\eta+1)}\left[\left(1-\frac{1-w}{2}\right)^r\right]^s|\Psi''(d)|^q+\left(\frac{1-w}{2}\right)^{rs}|\Psi''(c)|^q dw\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{(d-c)^2}{8(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times\\
&\left\{\left(\int_0^1(1-w)^{(\eta+1)}\frac{(2^r-(1-w)^r)^s}{2^{rs}}|\Psi''(c)|^q+\int_0^1\frac{(1-w)^{(\eta+1)}(1-w)^{rs}}{2^{rs}}|\Psi''(d)|\right)^{\frac{1}{q}}+\right.\\
&\left.\left(\int_0^1(1-w)^{(\eta+1)}\frac{(2^r-(1-w)^r)^s}{2^{rs}}|\Psi''(d)|^q+\int_0^1\frac{(1-w)^{(\eta+1)}(1-w)^{rs}}{2^{rs}}|\Psi''(c)|\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{(d-c)^2}{8(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2^{\frac{rs}{q}}}\right)\times\\
&\left\{\left(\int_0^1(1-w)^{(\eta+1)}[(2^r-(1-w)^r)^s]|\Psi''(c)|^q+\int_0^1(1-w)^{(\eta+1)}(1-w)^{rs}|\Psi''(d)|\right)^{\frac{1}{q}}+\right.\\
&\left.\left(\int_0^1(1-w)^{(\eta+1)}[(2^r-(1-w)^r)^s]|\Psi''(d)|^q+\int_0^1(1-w)^{(\eta+1)}(1-w)^{rs}|\Psi''(c)|\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{(d-c)^2}{2^{\frac{3q+rs}{q}}(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times\left\{\left(H(s,r,\eta,w)|\Psi''(c)|^q+\left(\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}\right)|\Psi''(d)|\right)^{\frac{1}{q}}\right.\\
&\left.+\left(H(s,r,\eta,w)|\Psi''(d)|^q+\left(\frac{\Gamma(rs+\eta+2)}{\Gamma(rs+\eta+3)}\right)|\Psi''(c)|\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

□

Corollary 3.9. *If we put $r = 1$ in Theorem 3.8, then (12) becomes*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$
$$\leq \frac{(d-c)^2}{2^{\frac{3q+s}{q}}(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times\left\{\left(H(s,\eta,w)|\Psi''(c)|^q+\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}|\Psi''(d)|\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(H(s,1,\eta,w)|\Psi''(d)|^q+\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}|\Psi''(c)|\right)^{\frac{1}{q}}\right\}$$

which is equal to Theorem 2.3 and Theorem 4 in [11].

Corollary 3.10. *If we put $s = 1$ in Theorem 3.8, then (12) becomes*

$$\left|\frac{2^{\eta+1}\Gamma(\eta+1)}{(d-c)^{\eta}}\left[J^{\eta}_{(\frac{c+d}{2})^-}\Psi(c)+J^{\eta}_{(\frac{c+d}{2})^+}\Psi(d)\right]-\Psi\left(\frac{c+d}{2}\right)\right|$$
$$\leq \frac{(d-c)^2}{8(\eta+1)}\left(\frac{1}{\eta+2}\right)^{1-\frac{1}{q}}\times$$
$$\left\{\left[|\Psi''(d)|^q\left(\frac{1}{\eta+1}-\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right)+\Psi''(c)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right]^{\frac{1}{q}}\right.$$
$$\left.\left[|\Psi''(c)|^q\left(\frac{1}{\eta+1}-\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right)+\Psi''(d)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\eta+2)}{\Gamma(s+\eta+3)}\right)\right]^{\frac{1}{q}}\right\}$$

which is equal to Theorem 2.6 and Theorem 7 in [11].

Remark 3.11.

(1) For $s = r = 1$ we will get all results for ordinary convex functions.

(2) For the results in which Hölder's inequality is used, if we put $p = q = 2$, then we get results corresponding to Cauchy-Schwarz inequality. Also, if we put $q = 1$ and let $p \to \infty$, then we get results for essentially bounded functions or results involving essential supremum.

(3) If we very values of η in our all main results we would get several new results. The case of $\eta = 1$ would be of special interest. Further we can also get many results as our special cases as we have done in our paper [1] and may capture results for many different classes of functions including class of quasi-convex function and class of $P-$convex functions etc.

4. Conclusion

Hermite−Hadamard dual inequality is one of the most celebrated inequalities. We can find its various generalizations and variants in literature. In [1] we have recently introduced a generalized notion of $(s,r)-$convex functions. This new class of functions contains many important classes including class of $s-$convex of the first and of the second kind (and hence contains class of convex functions). It also contains class of $P-$convex functions and class of quasi−convex functions under certain specific conditions. In this article, we have used this new notion of $(s,r)-$convex functions to find various bound of Hermite-Hadamard type functions. To be more specific, in Section 3, we have stated three different results related to estimation of bound of difference of right and middle term of Hermite−Hadamard type fractional integral dual inequality in absolute sense (similar to the estimation of trapezoidal rule). Here we used different techniques including Hölder's inequality and power mean inequality. Our proposed results captured various results stated in article [11].

Competing interests

The authors declare that they have no competing interests.

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Author's contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

References

[1] Bilal, M., Imtiaz, M., Khan, A. R., Khan, I. U., & Zafran., M. (2021). Generalized Hermite-Hadamard inequalities for $s-$convex functions in the mixed kind, (submitted).

[2] Breckner, W. W. (1978). Stetigkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topol-ogischen linearen Raumen. *Publications de l'Institut Mathmatique* **23**, 13-20.

[3] Cambini, A., & Martein, L. (2009). *Generalized convexity and optimization, Theory and applications, Lecture Notes in Economics and Mathematical System* **616** Springer.

[4] Cristescu, G., & Lupsa, L. (2002). *Non-connected Convexities and Applications.* Kluwer Academic Publishers, Dordrecht, Holland.

[5] Diewert, W. E., Avriel, M., & Zang, I. (1981). Nine senses of quasiconcavity and concavity. Reprint from *Journal of Economic Theory*, **25**(3), 397– 420.

[6] Dragomir, S. S., Pecaric, J., & Persson, L. E. (1995). Some inequalities of Hadamard type. *Soochow Journal of Mathematics* **21**(3), 335-341.

[7] Hermite, Ch. (1883). Sur deux limites d'une int'egrale d'efinie. *Mathesis* **3**, 82-83.

[8] Mihesan, V. A. (1993). Generalization of the convexity. *Seminar on Functional Equations, Approximation and Convexity*. Cluj-Napoca, Romania.

[9] Mitrinović, D. S., Pečarić, J. E., & Fink, A. M. (1993). *Classical and new inequaliites in analysis*. Klunwer, Academic Publishers, Dordrecht.

[10] Noor, M. A., Awan, M. U., & Noor, K. I. (2013). On some inequalities for relative semi-convex functions. *Journal of Inequalities and Applications*, **2013**, 332.

[11] Noor, M. A., & Awan, M. U. (2013). Some Integral inequalities for two kinds of convexities via fractional integrals. *Transylvanian Journal of Mathematics and Mechanics*, **5**(2), 129–136.

[12] Özcan, S., & Işcan, I. (2019). Some new Hermite-Hadamard type inequalities for $s-$convex functions and their applications. *Journal of Inequalitites and Applications*, **2019**, 201.

[13] Ozdemir, M. E., Yildiz, C. & Kavurmaci, H. (2012). Fractional integral inequalities for different functions. arXiv:1202.2431v1.

[14] Sarikaya, M. Z., Set, E., Yaldiz, H., & Basak, N. (2013). Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling* **57**, 2403-2407.

[15] Set, E., Özdemír, M. E., Sarikaya, M. Z. & Karakoc, F. (2015). Hermite–Hadamard type inequalities for mappings whose derivatives are $s-$convex in the second sense via fractional integrals. *Journal of Mathematics*, **1** (1), 62 – 70.

[16] Shuang, Y., Yin, H. P., & Qi, F. (2013). Hermite-Hadamard type integral inequalities for geometric-arithamatically $s-$convex functions. Analysis, **33**, 197-208.

[17] Toader, G., & Toader, S. (2009). A hierarchy of logarithmic convexity of functions. *Annals of the T. Popoviciu Seminar of Functional Equations, Approximation and Convexity*, **7**, 147–154.

[18] Varošanec, S. (2007). On h-convexity. *Journal of Mathematical Analysis and Applications*, **326**, 303-311.

[19] Wang, J., Li, X., Feckan, M., & Zhou, Y. (2013). Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Applicable Analysis, **92**(11), 2241-2253.

[20] Zhang, T. Y., Ji, A. P., & Qi, F. (2012). On integral inequalities of Hermite-Hadamard type for $s-$geometrically convex functions. *Abstract and Applied Analysis*, **2012**, Article ID 560586.