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# FEJÉR-HADAMARD TYPE INEQUALITIES FOR EXPONENTIALLY ($p$,$h$)-CONVEX FUNCTIONS VIA A GENERALIZED FRACTIONAL INTEGRAL

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**Abstract.** This paper includes new versions of generalized Fejér-Hadamard (FH) type fractional integral inequalities (FIIs) for exponentially ($p$,$h$)-convex functions. We utilize an extracted generalized fractional integral operator (FIO) involving Mittag-Leffler function (MLF) which contains a monotone increasing function. Some special already published results are deduced from the presented results.
**Keywords:** Exponentially ($p$,$h$)-convex functions, Fejér-Hadamard type inequalities, Generalized fractional integral operators, Mittag-Leffler function.

## 1. INTRODUCTION

In the study of mathematical inequalities, convex functions [22] play an important role. There have been studied a lot of inequalities for convex and related functions, see [1, 15, 18–21, 23, 25–27, 31]. In recent decades FIIs for convex and related functions are in focus of researchers. As a result many well known inequalities have been produced for various types of FIOs, see [1, 2, 11, 17, 18, 28]. The Hadamard (H) inequality is studied extensively for FIOs. The purpose of this paper is to introduce the weighted version (FH-inequality) of the H-inequality in the prospect of a generalized FIO. These fractional inequalities will hold simultaneously for convex, $p$-convex, $h$-convex, ($p$,$h$)-convex, exponentially $s$-convex functions.

Prabhaker presented the following integral operators defined in [24] containing MLF.

**Definition 1.1.** *Let $\phi:[a_0,b_0]\to\mathbb{R}$, $0<a_0<b_0$ be a function such that $\phi$ be positive and $\phi\in L_1[a_0,b_0]$. Also assume $\omega\in\mathbb{R}$ and $\rho,\sigma,\tau\in\mathbb{R}_+$. Then the FIOs involving MLF, $\varepsilon^{\rho}_{\sigma,\tau,\omega,a_0^+}\phi$ and $\varepsilon^{\rho}_{\sigma,\tau,\omega,b_0^-}\phi$ are defined by:*

$$\left(\varepsilon^{\rho}_{\sigma,\tau,\omega,a_0^+}\phi\right)(x)=\int_{a_0}^{x}(x-t)^{\tau-1}E^{\rho}_{\sigma,\tau}(\omega(x-t)^{\sigma})\phi(t)dt, \tag{1}$$

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$$\left(\varepsilon^{\rho}_{\sigma,\tau,\omega,b_0-}\phi\right)(x) = \int_x^{b_0}(t-x)^{\tau-1}E^{\rho}_{\sigma,\tau}(\omega(t-x)^{\sigma})\phi(t)dt, \tag{2}$$

*where* $E^{\rho}_{\sigma,\tau}(t)$ *is MLF;* $E^{\rho}_{\sigma,\tau}(t) = \sum_{n=0}^{\infty}\frac{(\rho)_n t^n}{\Gamma(\sigma n+\tau)n!}$ *where* $(\rho)_n$ *is the Pochhammer symbol defined as* $(\rho)_n = \rho(\rho+1)(\rho+2)...(\rho+n-1)$, $(\rho)_0=1$; $\Gamma(.)$ *is the gamma function defined as* $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx, n>0$; *also* $(\rho)_n = \frac{\Gamma(\rho+n)}{\Gamma(\rho)}$.

FIOs given by Andrić *et al.* in [2] linking with an extended generalized MLF in their kernels as follows:

**Definition 1.2.** *[2] Let* $\phi : [a_0,b_0] \to \mathbb{R}$, $0 < a_0 < b_0$ *be a function such that* $\phi$ *be positive and* $\phi \in L_1[a_0,b_0]$. *Also let* $\delta,\tau,c,\rho,\omega \in \mathbb{C}$ *and let* $\Re$ *be a real part of a complex number such that;* $\Re(\delta),\Re(\tau),\Re(\rho),\Re(c) > 0$, $\Re(\rho) < \Re(c)$ *with* $\sigma, r > 0$, $q \geq 0$ *and* $0 < k \leq \sigma + r$. *Then the generalized FIOs* $\varepsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a_0+}\phi$ *and* $\varepsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b_0-}\phi$ *are defined by:*

$$\left(\varepsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a_0+}\phi\right)(x;q) = \int_{a_0}^{x}(x-t)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(x-t)^{\sigma};q)\phi(t)dt, \tag{3}$$

$$\left(\varepsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b_0-}\phi\right)(x;q) = \int_{x}^{b_0}(t-x)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(t-x)^{\sigma};q)\phi(t)dt, \tag{4}$$

$$E^{\rho,r,k,c}_{\sigma,\tau,\delta}(t;q) = \sum_{n=0}^{\infty}\frac{\beta_q(\rho+nk,c-\rho)(c)_{nk}t^n}{\beta(\rho,c-\rho)\Gamma(\sigma n+\tau)(\delta)_{nr}}, \tag{5}$$

*is the extended generalized MLF and* $\beta_q$ *is the extension of* $\beta$ *function given as:*

$$\beta_q(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{q}{t(1-t)}}dt, \quad x,y,q \in \mathbb{R}_+.$$

Recently, a strenuous definition of a unified FIO by Farid is given in [6] (see, also [17]):

**Definition 1.3.** *Let* $\phi,\psi : [a_0,b_0] \to \mathbb{R}$, $0 < a_0 < b_0$ *be the functions such that; (i)* $\phi$ *be positive and* $\phi \in L_1[a_0,b_0]$ *and* $\psi$ *be a differentiable and strictly increasing. (ii) Also let* $\frac{\zeta}{x}$ *be an increasing function on* $[a_0,\infty)$ *and* $\delta,\tau,c,\rho,\omega \in \mathbb{C}$, $\Re(\delta),\Re(\tau),\Re(\rho),\Re(c) > 0$, $\Re(\rho) < \Re(c)$ *with* $\sigma,r > 0$, $q \geq 0$ *and* $0 < k \leq \sigma + r$. *Then for* $x \in [a_0,b_0]$ *the FIOs* $({}_{\psi}F^{\zeta,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a_0+}\phi)$ *and* $({}_{\psi}F^{\zeta,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b_0-}\phi)$ *are defined by:*

$$({}_{\psi}F^{\zeta,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a_0+}\phi)(x;q) = \int_{a_0}^{x}\frac{\zeta(\psi(x)-\psi(t))}{\psi(x)-\psi(t)}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(\psi(x)-\psi(t))^{\sigma};q)\phi(t)d(\psi(t)), \tag{6}$$

$$({}_{\psi}F^{\zeta,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b_0-}\phi)(x;q) = \int_{x}^{b_0}\frac{\zeta(\psi(t)-\psi(x))}{\psi(t)-\psi(x)}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(\psi(t)-\psi(x))^{\sigma};q)\phi(t)d(\psi(t)). \tag{7}$$

The following definition of generalized FIO containing extended MLF in the kernel can be extracted by taking $\zeta(x) = x^{\tau}$ in Definition 1.3.

**Definition 1.4.** *[26] Let* $\phi,\psi : [a_0,b_0] \to \mathbb{R}$, $0 < a_0 < b_0$ *be the functions such that; (i)* $\phi$ *be positive and* $\phi \in L_1[a_0,b_0]$ *and* $\psi$ *be a differentiable and strictly increasing. (ii) Also let* $\delta,c,\tau,\rho,\omega \in \mathbb{C}$, $\Re(\delta),\Re(\tau),\Re(\rho),\Re(c) > 0$, $\Re(\rho) < \Re(c)$ *with* $\sigma,r > 0$, $q \geq 0$ *and* $0 < k \leq \sigma + r$. *Then for* $x \in [a_0,b_0]$ *FIOs are defined by:*

$$\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a_0+}\phi\right)(x;q) = \int_{a_0}^{x}(\psi(x)-\psi(t))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(\psi(x)-\psi(t))^{\sigma};q)\phi(t)d(\psi(t)), \tag{8}$$

$$\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b_0-}\phi\right)(x;q) = \int_{x}^{b_0}(\psi(t)-\psi(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(\psi(t)-\psi(x))^{\sigma};q)\phi(t)d(\psi(t)). \tag{9}$$

The FIOs (8) and (9) produce particularly various known FIOs, see [27, Remark 1].

**Definition 1.5.** *[29] A function $\phi:[a_0,b_0]\to\mathbb{R}$ is said to be convex if*

$$\phi(\lambda x+(1-\lambda)y)\le\lambda\phi(x)+(1-\lambda)\phi(y),\ \lambda\in[0,1],\forall x,y\in[a_0,b_0]$$

*holds.*

The following H-inequality gives a necessary and sufficient definition for convex functions:

**Theorem 1.6.** *[9, 10] Let $\phi:[a_0,b_0]\to\mathbb{R}$ be a convex function such that $a_0<b_0$, then the following inequality holds:*

$$\phi\left(\frac{a_0+b_0}{2}\right)\le\frac{1}{b_0-a_0}\int_{a_0}^{b_0}\phi(x)dx\le\frac{\phi(a_0)+\phi(b_0)}{2}. \tag{10}$$

FH-inequality is a complete version of H-inequality given by Fejér:

**Theorem 1.7.** *[8] Let $\phi:[a_0,b_0]\to\mathbb{R}$ be a convex and $\eta:[a_0,b_0]\to\mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a_0+b_0}{2}$. Then inequality holds:*

$$\phi\left(\frac{a_0+b_0}{2}\right)\int_{a_0}^{b_0}\eta(x)dx\le\int_{a_0}^{b_0}\phi(x)\eta(x)dx\le\frac{\phi(a_0)+\phi(b_0)}{2}\int_{a_0}^{b_0}\eta(x)dx. \tag{11}$$

A notion of harmonically convex function is defined in [12].

**Definition 1.8.** *Let $I$ be an interval such that $I\subseteq\mathbb{R}_+$, then a function $\phi:I\to\mathbb{R}$ is said to be harmonically convex, if*

$$\phi\left(\frac{a_0b_0}{ta_0+(1-t)b_0}\right)\le t\phi(b_0)+(1-t)\phi(a_0)$$

*holds for all $a_0,b_0\in I$ and $t\in[0,1]$.*

Recently a lot of FIIs for instance H and FH-type for convex functions and related functions via FIOs have been published (see,[28] and references there in). Qiang *et al.* [26] gave new versions of H and FH-inequalities for harmonically convex functions for FIOs (8) and (9) as follows:

**Theorem 1.9.** *[26] Let $\phi,\psi:[a_0,b_0]\to\mathbb{R}$, $0<a_0<b_0$, Range $(\psi)\subset[a_0,b_0]$ be the functions such that $\phi$ be positive and $\phi\in L_1[a_0,b_0]$ and $\psi$ be a differentiable and strictly increasing. If $\phi$ is a harmonically convex function on $[a_0,b_0]$, then for FIOs (8) and (9) we have*

$$\begin{aligned}
&\phi\left(\frac{2\psi(a_0)\psi(b_0)}{\psi(a_0)+\psi(b_0)}\right)\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right)\right)^-}1\right)\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right);q\right)\\
&\le\frac{1}{2}\left(\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right)\right)^-}\phi\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right);q\right)\right.\\
&\left.+\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right)\right)^+}\phi\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right);q\right)\right)\\
&\le\frac{\phi(\psi(a_0))+\phi(\psi(b_0))}{2}\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right)\right)^+}1\right)\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right);q\right),
\end{aligned}$$

*where $\xi(t)=\frac{1}{\psi(t)}$ for all $t\in[\frac{1}{b_0},\frac{1}{a_0}]$ and $\omega_0=\omega\left(\frac{\psi(a_0)(b_0)}{\psi(b_0)-\psi(a_0)}\right)^{\sigma}$.*

**Theorem 1.10.** *[26] Let $\phi,\psi,\eta:[a_0,b_0]\to\mathbb{R}$, $0<a_0<b_0$, Range $(\psi)$, Range $(\eta)\subset[a_0,b_0]$ be the functions such that $\phi$ be positive and $\phi\in L_1[a_0,b_0]$, $\psi$ be a differentiable and strictly increasing and $\eta$ be a non-negative and integrable. If $\phi(\frac{1}{\psi(x)})=\phi\left(\frac{1}{\frac{1}{\psi(a_0)}+\frac{1}{\psi(b_0)}-\psi(x)}\right)$ then for FIOs (8) and (9) we have:*

$$\begin{aligned}
&\phi\left(\frac{2\psi(a_0)\psi(b_0)}{\psi(a_0)+\psi(b_0)}\right)\left(\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right)\right)^{+}}\eta\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right);q\right)\right.\\
&\left.+\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right)\right)^{-}}\eta\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right);q\right)\right)\\
&\le\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right)\right)^{+}}\phi\eta\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right);q\right)\\
&+\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right)\right)^{-}}\phi\eta\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right);q\right)\\
&\le\frac{\phi(\psi(a_0))+\phi(\psi(b_0))}{2}\left(\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right)\right)^{+}}\eta\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right);q\right)\right.\\
&\left.+\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega_0,\left(\psi^{-1}\left(\frac{1}{\psi(a_0)}\right)\right)^{-}}\eta\circ\xi\right)\left(\psi^{-1}\left(\frac{1}{\psi(b_0)}\right);q\right)\right),
\end{aligned}\tag{12}$$

*where $\xi(t)=\frac{1}{\psi(t)}$ for all $t\in[\frac{1}{b_0},\frac{1}{a_0}]$, $\phi\eta\circ\xi=(\phi\circ\xi)(\eta\circ\xi)$ and $\omega_0=\omega(\frac{\psi(a_0)\psi(b_0)}{\psi(b_0)-\psi(a_0)})^{\sigma}$.*

**Definition 1.11.** *[30] A positive real-valued function $\phi:\mathfrak{J}\subset\mathbb{R}\to\mathbb{R}_+$ is said to be exponentially convex on $\mathfrak{J}$ if the inequality*

$$\phi(\lambda x+(1-\lambda)y)\le\lambda\frac{\phi(x)}{e^{\alpha x}}+(1-\lambda)\frac{\phi(y)}{e^{\alpha y}},\ \lambda\in[0,1],\forall x,y\in\mathbb{R},\alpha\in\mathbb{R}$$

*holds.*

**Definition 1.12.** *[13] Let $\mathfrak{J}\subset\mathbb{R}_+$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. Then a function $\phi:\mathfrak{J}\to\mathbb{R}$ is said to be p-convex, if*

$$\phi\left((\lambda a_0{}^p+(1-\lambda)b_0{}^p)^{\frac{1}{p}}\right)\le\lambda\phi(a_0)+(1-\lambda)\phi(b_0)$$

*holds for $a_0,b_0\in\mathfrak{J}$ and $\lambda\in[0,1]$.*

**Definition 1.13.** *[5] Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. Let $\mathfrak{J}\subset\mathbb{R}_+$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. A function $\phi:\mathfrak{J}\to\mathbb{R}$ is said to be (p,h)-convex, if*

$$\phi\left((\lambda a_0{}^p+(1-\lambda)b_0{}^p)^{\frac{1}{p}}\right)\le h(\lambda)\phi(a_0)+h(1-\lambda)\phi(b_0)\tag{13}$$

*holds for $a_0,b_0\in\mathfrak{J}$ and $\lambda\in[0,1]$.*

If the inequality in (13) is reversed, then $\phi$ is called $(p,h)$-concave.

**Definition 1.14.** *[21] Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. Let $\mathfrak{J}\subset\mathbb{R}_+$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. A function $\phi:\mathfrak{J}\to\mathbb{R}$ is said to be exponentially (p,h)-convex, if*

$$\phi\left((\lambda a_0{}^p+(1-\lambda)b_0{}^p)^{\frac{1}{p}}\right)\le h(\lambda)\frac{\phi(a_0)}{e^{\alpha a_0}}+h(1-\lambda)\frac{\phi(b_0)}{e^{\alpha b_0}}\tag{14}$$

*holds for $a_0,b_0\in\mathfrak{J}$, $\alpha\in\mathbb{R}$ and $\lambda\in[0,1]$.*

By reversing the inequality in (14), the exponentially ($p$,$h$)-concave function is obtained. On fixing $\alpha = 0$, an exponentially ($p$,$h$)-convex function takes the form of ($p$,$h$)-convex function.

Upcoming section 2 contains two versions of FH-inequality for exponentially ($p$,$h$)-convex functions in the span of generalized FIOs (8) and (9).

In the whole paper we use the following notations as follows:
$\left(\mathscr{F}_{b_0,\tau,\sigma}^{a_0^+}\right)(\omega,\phi;q)=\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a_0^+}\phi\right)(b_0;q)$, $\left(\mathscr{F}_{a_0,\tau,\sigma}^{b_0^-}\right)(\omega,\phi;q)=\left({}_{\psi}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b_0^-}\phi\right)(a_0;q)$.

## 2. Fejér-Hadamard type inequalities

**Theorem 2.1.** *Let $\phi,\eta,\psi:[a_0,b_0]\to\mathbb{R}$, $0<a_0<b_0$, Range $(\eta)$, Range $(\psi)\subset[a_0,b_0]$ be the functions such that $\phi$ and $\eta$ be positive, $\phi$ and $\eta\in L_1[a_0,b_0]$, where $\psi$ be a differentiable and strictly increasing. If $\phi$ is exponentially (p,h)-convex and $\phi(\psi(x))=\phi(\psi^p(a_0)+\psi^p(b_0)-\psi(x))$ and $\eta$ is symmetric about $\frac{\psi^p(a_0)+\psi^p(b_0)}{2}$ then the following inequalities for FIOs (8) and (9) hold:*

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^{\frac{1}{p}}\right)\mathfrak{D}(\alpha)\left(\mathscr{F}^{\psi^{-1}(\psi^p(a_0))^+}_{\psi^{-1}(\psi^p(b_0)),\tau,\sigma}\right)(\omega_0,\eta\circ\xi;q)\\
&\leq\left(\mathscr{F}^{\psi^{-1}(\psi^p(a_0))^+}_{\psi^{-1}(\psi^p(b_0)),\tau,\sigma}\right)(\omega_0,\phi\eta\circ\xi;q)+\left(\mathscr{F}^{\psi^{-1}(\psi^p(b_0))^-}_{\psi^{-1}(\psi^p(a_0)),\tau,\sigma}\right)(\omega_0,\phi\eta\circ\xi;q)\\
&\leq\left(\frac{\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}+\frac{\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}\right)\\
&\times\int_0^1 t^{\tau-1}(h(t)+h(1-t))\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)dt,
\end{aligned}\tag{15}$$

*where $\xi(t)=\psi^{\frac{1}{p}}(t)$, $\mathfrak{D}(\alpha)=e^{\alpha\psi(b_0)}$ for $\alpha<0$, $\mathfrak{D}(\alpha)=e^{\alpha\psi(a_0)}$ for $\alpha\geq 0$, $\phi\eta\circ\xi=(f\circ\xi)(\eta\circ\xi)$ and $\omega_0=\frac{\omega}{(\psi^p(b_0)-\psi^p(a_0))^{\sigma}}$.*

*Proof.* By exponentially ($p$,$h$)-convexity of $\phi$, one can observe

$$\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(x)+\psi^p(y)}{2}\right)^{\frac{1}{p}}\right)\leq\frac{\phi(\psi(x))}{e^{\alpha\psi(x)}}+\frac{\phi(\psi(y))}{e^{\alpha\psi(y)}}.\tag{16}$$

On setting $\psi(x)=(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}$ and $\psi(y)=(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}$ in (16) we get

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^{\frac{1}{p}}\right)\\
&\leq\frac{\phi\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}}}+\frac{\phi\left((t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}}}.
\end{aligned}\tag{17}$$

Multiplying (17) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)$ and integrating over $[0,1]$, we get

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^{\frac{1}{p}}\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)dt\\
&\leq \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\phi\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}}}dt\\
&+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\phi\left((t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}\right)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}}}dt.
\end{aligned} \tag{18}$$

Setting $t\psi^p(a_0)+(1-t)\psi^p(b_0)=\psi(x)$, that is $((1-t)\psi^p(a_0)+t\psi^p(b_0)=\psi^p(a_0)+\psi^p(b_0)-\psi(x)$ in (18) and utilizing condition $\phi(\psi(x))=\phi(\psi^p(a_0)+\psi^p(b_0)-\psi(x))$, then by FIOs (8) and (9) one can get first inequality of (15).

Again by using the exponentially $(p,h)$-convexity of $\phi$, the right hand side of (17) leads

$$\begin{aligned}
&\frac{\phi\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}}}+\frac{\phi\left((t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}}}\\
&\leq\frac{1}{e^{\alpha(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}}}\left(\frac{h(t)\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}+\frac{h(1-t)\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}\right)\\
&+\frac{1}{e^{\alpha(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}}}\left(\frac{h(t)\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}+\frac{h(1-t)\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}\right).
\end{aligned} \tag{19}$$

Now, multiplying (19) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\phi\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}}}dt\\
&+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\phi\left((t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}\right)\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}}}dt\\
&\le\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}}}\left(\frac{h(t)\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}+\frac{h(1-t)\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}\right)dt\\
&+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\eta\left((t\psi^p(a_0)+(1-t)\psi^p(b_0))^{\frac{1}{p}}\right)}{e^{\alpha(t\psi^p(b_0)+(1-t)\psi^p(a_0))^{\frac{1}{p}}}}\left(\frac{h(t)\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}+\frac{h(1-t)\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}\right)dt. \qquad (20)
\end{aligned}$$

Again setting $t\psi^p(a_0)+(1-t)\psi^p(b_0)=\psi(x)$, that is $((1-t)\psi^p(a_0)+t\psi^p(b_0)=\psi^p(a_0)+\psi^p(b_0)-\psi(x)$ in (20) and utilizing condition $\phi(\psi(x))=\phi(\psi^p(a_0)+\psi^p(b_0)-\psi(x))$, then by FIOs (8) and (9), one can attain second inequality of (15). □

**Remark 2.2.**

*(1) On fixing $q=0$, $\psi=I$, $\alpha=0$, $\eta(x)=1$, $h(t)=t$ and $p=-1$ in Theorem 2.1, [3, Theorem 3.1] is recovered.*

*(2) On fixing $\eta(x)=1$, $\alpha=0$, $h(t)=t$, $p=-1$ and $\psi=I$ in Theorem 2.1, [7, Theorem 2.1] is recovered.*

*(3) On fixing $\omega=q=0$, $\alpha=0$, $\eta(x)=1$, $p=-1$, $h(t)=t$ and $\psi=I$ in Theorem 2.1, [14, Theorem 4] is recovered.*

*(4) On fixing $\omega=0$, $h(t)=t$, $\alpha=0$, $\tau=1$, $q=0$, $\psi=I$ and $p=-1$ in Theorem 2.1, [4, Theorem 8] is recovered.*

*(5) On fixing $\omega=0$, $p=-1$, $q=0$, $\alpha=0$, $\tau=1$, $\psi=I$, $h(t)=t$ and $\eta(x)=1$ in Theorem 2.1, [16, Theorem 2.4] is recovered.*

*(6) On fixing $\alpha=0$, $p=-1$ and $h(t)=t$ in Theorem 2.1, [26, Theorem 2.5] is recovered.*

*(7) On fixing $\alpha=0$, $h(t)=t$ and $p=-1$ in Theorem 2.1, Theorem 1.10 is recovered.*

Now we give another FH-inequality:

**Theorem 2.3.** *Let $\phi,\eta,\psi:[a_0,b_0]\to\mathbb{R}$, $0<a_0<b_0$, Range $(\eta)$, Range$(\psi)\subset[a_0,b_0]$ be the functions such that $\phi$ and $\eta$ be positive, $\phi$ and $\eta\in L_1[a_0,b_0]$ and $\psi$ be a differentiable and strictly increasing. If $\phi$ is exponentially (p,h)-convex and $\phi(\psi(x))=\phi(\psi^p(a_0)+\psi^p(b_0)-\psi(x))$ and $\eta$ is symmetric about $\frac{\psi^p(a_0)+\psi^p(b_0)}{2}$ then the*

*following inequalities for FIOs (8) and (9) hold:*

$$
\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^{\frac{1}{p}}\right)\mathfrak{D}(\alpha)\left(\mathscr{F}^{\psi^{-1}\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^+}_{\psi^{-1}(\psi^p(b_0)),\tau,\sigma}\right)(2^\sigma\omega_0,\eta\circ\xi;q)\\
&\leq\left(\mathscr{F}^{\psi^{-1}\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^+}_{\psi^{-1}(\psi^p(b_0)),\tau,\sigma}\right)(2^\sigma\omega_0,\phi\eta\circ\xi;q)+\left(\mathscr{F}^{\psi^{-1}\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^-}_{\psi^{-1}(\psi^p(a_0)),\tau,\sigma}\right)(2^\sigma\omega_0,\phi\eta\circ\xi;q)\\
&\leq\left(\frac{\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}+\frac{\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}\right)\\
&\times\int_0^1 t^{\tau-1}\left(h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right)\\
&\times\eta\left(\left(\frac{t}{2}\psi^p(a_0)+\left(1-\frac{t}{2}\right)\psi^p(b_0)\right)^{\frac{1}{p}}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;q)dt,
\end{aligned}
\tag{21}
$$

*where* $\xi(t)=\psi^{\frac{1}{p}}(t)$, $\mathfrak{D}(\alpha)=e^{\alpha\psi(b_0)}$ *for* $\alpha<0$, $\mathfrak{D}(\alpha)=e^{\alpha\psi(a_0)}$ *for* $\alpha\geq 0$, $\phi\eta\circ\xi=(\phi\circ\xi)(\eta\circ\xi)$ *and* $\omega_0=\frac{\omega}{(\psi^p(b_0)-\psi^p(a_0))^\sigma}$.

*Proof.* By setting $\psi(x)=\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}$, $\psi(y)=\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}$ in (16) we get

$$
\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^{\frac{1}{p}}\right)\\
&\leq\frac{\phi\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}}}+\frac{\phi\left(\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}}}.
\end{aligned}
\tag{22}
$$

Multiplying (22) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\eta\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)$ and integrating over $[0,1]$, we get

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\phi\left(\left(\frac{\psi^p(a_0)+\psi^p(b_0)}{2}\right)^{\frac{1}{p}}\right)\\
&\times\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\eta\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)dt\\
&\leq\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\phi\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)\eta\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}}}dt\\
&+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\\
&\times\frac{\phi\left(\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}\right)\eta\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}}}dt.
\end{aligned}\tag{23}$$

Setting $\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)=\psi(x)$, that is $(1-\frac{t}{2})\psi^p(a_0)+\frac{t}{2}\psi^p(b_0)=\psi^p(a_0)+\psi^p(b_0)-\psi(x)$ in (23) and employing condition $\phi(\psi(x))=\phi(\psi^p(a_0)+\psi^p(b_0)-\psi(x))$, then by FIOs (8) and (9), one can get first inequality of (21).

Again by using the exponentially $(p,h)$-convexity of $\phi$, the right hand side of (22) leads

$$\begin{aligned}
&\frac{\phi\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}}}+\frac{\phi\left(\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}}}\\
&\leq\frac{1}{e^{\alpha\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}}}\left(\frac{h(\frac{t}{2})\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}+\frac{h(1-\frac{t}{2})\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}\right)\\
&+\frac{1}{e^{\alpha\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}}}\left(\frac{h(\frac{t}{2})\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}+\frac{h(1-\frac{t}{2})\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}\right).
\end{aligned}\tag{24}$$

Multiplying both sides of (24) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\frac{\phi\left(\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}}}dt \qquad (25)\\
&+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\frac{\phi\left(\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}\right)}{e^{\alpha\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}}}dt\\
&\leq \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\frac{1}{e^{\alpha\left(\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)\right)^{\frac{1}{p}}}}\\
&\times\left(\frac{h\left(\frac{t}{2}\right)\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}+\frac{h\left(1-\frac{t}{2}\right)\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}\right)dt\\
&+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};q)\frac{1}{e^{\alpha\left(\frac{t}{2}\psi^p(b_0)+(1-\frac{t}{2})\psi^p(a_0)\right)^{\frac{1}{p}}}}\\
&\times\left(\frac{h\left(\frac{t}{2}\right)\phi(\psi(b_0))}{e^{\alpha\psi(b_0)}}+\frac{h\left(1-\frac{t}{2}\right)\phi(\psi(a_0))}{e^{\alpha\psi(a_0)}}\right)dt.
\end{aligned}$$

Again setting $\frac{t}{2}\psi^p(a_0)+(1-\frac{t}{2})\psi^p(b_0)=\psi(x)$, that is $(1-\frac{t}{2})\psi^p(a_0)+\frac{t}{2}\psi^p(b_0)=\psi^p(a_0)+\psi^p(b_0)-\psi(x)$ in (25) and employing condition $\phi(\psi(x))=\phi(\psi^p(a_0)+\psi^p(b_0)-\psi(x))$, then by FIOs (8) and (9), one can attain second inequality of (21).

**Remark 2.4.**

*(1) On fixing $h(t)=t$, $q=0$, $\psi=I$ and $p=-1$ in Theorem 2.3, [3, Theorem 3.6 ] is recovered.*
*(2) On fixing $h(t)=t$, $p=-1$ and $\psi=I$ in Theorem 2.3, [7, Theorem 2.6 ] is recovered.*
*(3) On fixing $\alpha=0$, $p=-1$ and $h(t)=t$ in Theorem 2.3, [26, Theorem 2.10] is obtained.*
*(4) From Theorems 2.1-2.3, one can find results for the operators given in [27, Remark 1].*
*(5) On fixing $\eta(x)=1$ in Theorems 2.1-2.3, results for the H-inequality can be attained.*
*(6) On fixing $h(t)=t$ in Theorems 2.1-2.3, results for exponentially p-convex functions can be attained.*

□

## COMPETING INTERESTS

The authors declare that they have no competing interests.

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## AUTHOR'S CONTRIBUTIONS

All authors equally contributed to this work. All authors read and approved the final manuscript.

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