ISSN 2709-4170 (Print)
https://doi.org/10.52587/TMCS010204
Transactions in Mathematical and Computational Sciences
Vol. 1(2)(2021) pp. 31–50

# SOME GENERALIZED HADAMARD TYPE TRAPEZOID INEQUALITIES VIA EXPONENTIALLY $(m_1, m_2)-$ CONVEX FUNCTIONS AND THE FRACTIONAL INTEGRAL OPERATOR

BAHTIYAR BAYRAKTAR[1,*], SAAD IHSAN BUTT[2], SHAHID SHAOKAT[2]

[1]Bursa Uludag University, Department of Mathematics and Science Education, 16059, Bursa, Turkey
[2]COMSAT University Islamabad, Department of Mathematics, Lahore Campus, Pakistan

---

**Abstract.** In this paper, on the basis of the proved identities, we obtain generalized integral inequalities of the Hermite-Hadamard type for exponentially $(m_1, m_2)-$convex functions in terms of the Riemann-Liouville fractional integration operators. Some results existing in the literature are some special cases of our results.

**Keywords:** exponentially convex functions, exponentially $(m_1, m_2)-$ convex functions, Hermite–Hadamard inequality, Hölder inequality, Hölder- Işcan inequality, Improved power-mean inequality, Riemann–Liouville fractional integrals.

## 1. Introduction

The study of convex functions always offers and provide an amazing and excellent glimpse of the beauty and fascination of advance mathematics. The mathematicians always put an ability and working in this direction as a result, find and investigate have a large variety of results that have fruitful and remarkable applications. The study of convex functions has attracted shouldint a big deal of attention not only from the mathematicians, but also from people working various other fields such as economics, statistics, data mining, physics and signal processing. The theory of convexity also played amazing role in the development of theory of inequalities. In the last decades, many mathematician have worked on inequalities in the area of mathematics especially mathematical analysis, mathematical physics and functional analysis. No one can refuse its importance and significance and with the upcoming times this field is going on to robustness and widespread. Inequalities have many uses in probability and statistical problems. Due to paramount background,

*Correspondence: bbayraktar@uludag.edu.tr
Received: 14.03.2021 Revised: 02.02.2021 Accepted: 28.12.2021 Published: 31.12.2021

convex analysis and inequalities have become an absorbing field for the attention of mathematicians and readers, see [1–7] and references in them.
The following definitions are well known in the literature.

**Definition 1.** The function $g:[a_1,a_2]\to\mathbb{R}$, is said to be convex, if we have

$$g(\lambda x+(1-\lambda)y)\leq\lambda g(x)+(1-\lambda)g(y)$$

for all $x,y\in[a_1,a_2]$ and $\lambda\in[0,1]$.

**Definition 2.** Let $g:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and let $a_1,a_2\in I$, with $a_1<a_2$. The following double inequality;

$$g\left(\frac{a_1+a_2}{2}\right)\leq\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\leq\frac{g(a_1)+g(a_2)}{2}. \tag{1}$$

is known in the literature as Hadamard's inequality. If $g$ is concave then both inequalities hold in the reversed direction.

**Definition 3.** ([8])For $(m_1,m_2)\in(0,1]^2$, a function $g:[0,a_2]\to\mathbb{R}$ is said to be $(m_1,m_2)-$convex, if

$$g(m_1\xi x+m_2(1-\xi)y)\leq m_1\xi g(x)+m_2(1-\xi)g(y)$$

for all $x,y\in[0,a_2]$ and $\xi\in[0,1]$.

In [9], Awan et al. gave the following definition of exponentially convex function.

**Definition 4.** ([9]) A mapping $g:I\subseteq\mathbb{R}\to\mathbb{R}$ is called exponentially convex, if

$$g(\xi a_1+(1-\xi)a_2)\leq\xi\,\frac{g(a_1)}{e^{\lambda a_1}}\;+\;(1-\xi)\,\frac{g(a_2)}{e^{\lambda a_2}}, \tag{2}$$

for all $a_1,a_2\in I$, $\xi\in[0,1]$ and $\lambda\in\mathbb{R}$. Equation (2) is reversed, then $g$ is called exponentially concave.

Now, we give the definition of a exponentially $(m_1,m_2)-$convex [10].

**Definition 5.** For $(m_1,m_2)\in(0,1]^2$ and $\lambda\in\mathbb{R}$, a function $g:[0,a_2]\to\mathbb{R}$, is said to be exponentially $(m_1,m_2)-$convex, if

$$g(m_1\xi x+m_2(1-\xi)y)\leq m_1\xi\frac{g(x)}{e^{\lambda x}}+m_2(1-\xi)\frac{g(y)}{e^{\lambda y}} \tag{3}$$

for all $x,y\in[0,a_2]$ and $\xi\in[0,1]$.

**Remark 1.1.** *It follows from Definition 5 that:*

1. *if* $m_1 = m_2 = 1$ *and* $\lambda = 0$ *then* $g(x)$ *is an classical convex function on* $[0, a_2]$;
2. *if* $\lambda = 0$, *then* $g(x)$ *is an* $(m_1, m_2) -$ *convex function on* $[0, a_2]$;
3. *if* $\lambda = 0$ *and* $m_1 = 1$ *and* $m = m_2$, *then* $g(x)$ *is an* $m -$ *convex function on* $[0, a_2]$;
4. *if* $m_1 = m_2 = 1$, *then* $g(x)$ *is an exponentially convex function on* $[0, a_2]$;
5. *if* $m_1 = 1$ *and* $m = m_2$ *then* $g(x)$ *is an exponentially* $m -$ *convex function on* $[0, a_2]$

The classical definition of a Riemann–Liouville fractional integral in the literature is given in the following way

**Definition 6.** Let $g \in L_1[a_1, a_2]$. The Riemann Liouville integrals $J^{\alpha}_{a_1^+} g$ and $J^{\alpha}_{a_2^-} g$ of order $\alpha > 0$ with $a_1 \geq 0$ are defined by

$$J^{\alpha}_{a_1^+} g\ (x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x - \xi)^{\alpha - 1} g(\xi)\, d\xi\ , \qquad x > a_1$$

and

$$J^{\alpha}_{a_2 -} g\ (x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{a_2} (\xi - x)^{\alpha - 1} g(\xi)\, d\xi\ , \qquad x < a_2$$

respectively where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u} u^{\alpha - 1} du$. Here is $J^{0}_{a_1^+} g(x) = J^{0}_{a_2^-} g(x) = g(x)$. In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

The refinement of Hölder's integral inequality is given as follows:

**Theorem 1.2.** *(Hölder−Işcan Integral Inequality [11]) Let* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $u$ *and* $g$ *are real functions defined on* $[a_1, a_2]$ *and if* $|u|^p$, $|g|^q$ *are integrable functions on* $[a_1, a_2]$ *then*

$$\int_{a_1}^{a_2} |u(\xi) g(\xi)|\, d\xi \leq \frac{1}{a_2 - a_1} \left\{ \left( \int_{a_1}^{a_2} (a_2 - v)\, |u(\xi)|^p\, dx \right)^{\frac{1}{p}} \left( \int_{a_1}^{a_2} (a_2 - \xi)\, |g(\xi)|^q\, d\xi \right)^{\frac{1}{q}} \right.$$
$$\left. + \left( \int_{a_1}^{a_2} (x - a_1)\, |u(\xi)|^p\, d\xi \right)^{\frac{1}{p}} \left( \int_{a_1}^{a_2} (\xi - a_1)\, |g(\xi)|^q\, d\xi \right)^{\frac{1}{q}} \right\}.$$

An improved form of the power mean inequality (another form of the previous inequality), is given by the following theorem:

**Theorem 1.3.** *(Improved power-mean integral inequality [12]). Let $q \geq 1$. If $u$ and $g$ are real functions defined on $[a_1, a_2]$ and if $|u|$, $|u||g|^q$ are integrable functions on $[a_1, a_2]$ then*

$$\int_{a_1}^{a_2} |u(\xi)g(\xi)|\,d\xi \leq \frac{1}{a_2-a_1}\left\{\left(\int_{a_1}^{a_2}(a_2-\xi)\,|u(\xi)|\,d\xi\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_{a_1}^{a_2}(a_2-\xi)\,|u(\xi)|\,|g(\xi)|^q\,d\xi\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{a_1}^{a_2}(\xi-a_1)\,|u(\xi)|\,d\xi\right)^{1-\frac{1}{q}}\left(\int_{a_1}^{a_2}(\xi-a_1)\,|u(\xi)|\,|g(\xi)|^q\,d\xi\right)^{\frac{1}{q}}\right\}.$$

The purpose of the article is to obtain generalized integral inequalities of the Hadamard type for exponentially $(m_1, m_2)-$convex functions. These results are obtained using the classical Hölder inequalities and the improved Hölder-Işcan inequality.

## 2. Main Results

The Lemma stated below is a generalization of Lemma 3.1 from [13].

**Lemma 2.1.** *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on $I^\circ$. If $a_1m_1, a_2m_2 \in I$, with $a_1m_1 \neq a_2m_2$ and $g'' \in L[m_1a_1, a_2m_2]$, then for all $\alpha > 1$ the following equality holds*

$$\frac{g(m_1a_1)+g(m_2a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(m_2a_2-a_1m_1)^{\alpha-1}} \times U = \frac{(m_2a_2-a_1m_1)^2}{2}\,(I_1+I_2)\,, \tag{4}$$

*where*

$$U = \frac{(\alpha+1)}{m_2a_2-a_1m_1}\left[J^{\alpha}_{a_1m_1^+}g(a_2m_2)+J^{\alpha}_{a_2m_2^-}g(a_1m_1)\right] - \left[J^{\alpha-1}_{a_1m_1^+}g(a_2m_2)+J^{\alpha-1}_{a_2m_2^-}g(a_1m_1)\right],$$

$$I_1 = \int_0^1 \xi(1-\xi)^\alpha g''(a_1m_1\xi+(1-\xi)a_2m_2)d\xi$$

*and*

$$I_2 = \int_0^1 \xi(1-\xi)^\alpha g''((1-\xi)a_1m_1+\xi a_2m_2)d\xi.$$

*Proof.* To calculate the integrals first we make a transformation of variables $1-\xi=s$, and then twice integrating by parts we obtain:

$$I_1=\int_0^1 s^{\alpha}(1-s)g''((1-s)a_1m_1+sa_2m_2)ds=\frac{g(a_2m_2)}{(a_2m_2-a_1m_1)^2}$$
$$+\frac{\alpha(\alpha-1)}{(a_2m_2-a_1m_1)^2}\int_0^1 s^{\alpha-2}g((1-s)a_1m_1+sa_2m_2)ds$$
$$-\frac{\alpha(\alpha+1)}{(a_2m_2-a_1m_1)^2}\int_0^1 s^{\alpha-1}g((1-s)a_1m_1+sa_2m_2)ds$$

If we make $(1-s)a_1m_1+sa_2m_2=x$ the transformation in the both integrals obtained and taking into account the property of the Gamma functions, we obtain:

$$\begin{aligned} I_1 &= \frac{g(a_2m_2)}{(a_2m_2-a_1m_1)^2}+\frac{\Gamma(\alpha+1)}{(a_2m_2-a_1m_1)^{\alpha+1}}J^{\alpha-1}_{a_2m_2^-}g(a_1m_1)\\ &-\frac{\Gamma(\alpha+2)}{(a_2m_2-a_1m_1)^{\alpha+2}}J^{\alpha}_{a_2m_2^-}g(a_1m_1)\end{aligned}$$

Similarly for $I_2$, we have

$$\begin{aligned} I_2 &= \frac{g(a_1m_1)}{(a_2m_2-a_1m_1)^2}+\frac{\Gamma(\alpha+1)}{(a_2m_2-a_1m_1)^{\alpha+1}}J^{\alpha-1}_{a_1m_1^+}g(a_2m_2)\\ &-\frac{\Gamma(\alpha+2)}{(a_2m_2-a_1m_1)^{\alpha+2}}J^{\alpha}_{a_1m_1^+}g(a_2m_2)\end{aligned}$$

Summing these equalities and then grouping the summands we get

$$I_1+I_2=\frac{1}{(a_2m_2-a_1m_1)^2}[g(a_1m_1)+g(a_2m_2)]-\frac{\Gamma(\alpha+1)}{(a_2m_2-a_1m_1)^{\alpha+1}}\times U \tag{5}$$

Finally multiplying both side of the equality (5) by the expression $\frac{(a_2m_2-a_1m_1)^2}{2}$ we obtain (4). Hence, proof is completely done □

**Remark 2.2.** *If we take $m_1=m_2=1$, then from Lemma 2.1 we obtain the equality known in [13] (see Lemma 3.1).*

**Theorem 2.3.** *Let $g:I=[0,a_2^*]\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on $I^\circ$. If $a_1,a_2\in I$, $g''\in L[0,a_2^*]$, $g''$ positively defined and $|g''|$ is a exponentialy $(m_1,m_2)-$ convex function, whith $m_1,m_2\in(0,1]$, $\lambda\leq 0$, then for all $\alpha>1$ the following inequality holds*

$$\begin{aligned} &\left|\frac{g(m_1a_1)+g(m_2a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}}\times U\right| \\ \leq\ &\frac{(m_2a_2-m_1a_1)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{m_1|g''(a_1)|}{e^{\lambda a_1}}+\frac{m_2|g''(a_2)|}{e^{\lambda a_2}}\right],\end{aligned} \tag{6}$$

*where*

$$U = \frac{(\alpha+1)}{m_2a_2 - m_1a_1}\left[J^{\alpha}_{m_1a_1^+}g(m_2a_2) + J^{\alpha}_{m_2a_2^-}g(m_1a_1)\right] - \left[J^{\alpha-1}_{m_1a_1^+}g(m_2a_2) + J^{\alpha-1}_{m_2a_2^-}g(m_1a_1)\right],$$

*Proof.* From Lemma 2.1 and the properties of the module we have:

$$\left|\frac{g(m_1a_1)+g(m_2a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}} \times U\right| \leq \frac{(m_2a_2-m_1a_1)^2}{2}\left(|I_1|+|I_2|\right). \tag{7}$$

Considering the fact that $g''$ is an exponentially $(m_1,m_2)-$convex function, for the first integral we get:

$$\begin{aligned} |I_1| &\leq \int_0^1 \xi(1-\xi)^{\alpha}\left|g''(m_1a_1\xi + (1-\xi)m_2a_2)\right| d\xi \\ &\leq \frac{m_1|g''(a_1)|}{e^{\lambda a_1}}\int_0^1 \xi^2(1-\xi)^{\alpha}d\xi + \frac{m_2|g''(a_2)|}{e^{\lambda a_2}}\int_0^1 \xi(1-\xi)^{\alpha+1}d\xi \\ &= \frac{2m_1|g''(a_1)|}{(\alpha+1)(\alpha+2)(\alpha+3)e^{\lambda a_1}} + \frac{m_2|g''(a_2)|}{(\alpha+2)(\alpha+3)e^{\lambda a_2}}. \end{aligned} \tag{8}$$

Similarly, for the second integral we get:

$$|I_2| \leq \frac{m_1|g''(a_1)|}{(\alpha+2)(\alpha+3)e^{\lambda a_1}} + \frac{2m_2|g''(a_2)|}{(\alpha+1)(\alpha+2)(\alpha+3)e^{\lambda a_2}}. \tag{9}$$

Adding inequalities (8) and (9) we get:

$$|I_1|+|I_2| \leq \frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{m_1|g''(a_1)|}{e^{\lambda a_1}} + \frac{m_2|g''(a_2)|}{e^{\lambda a_2}}\right]. \tag{10}$$

Inequality (6) follows from (7) and (10). The proof is completed. □

**Corollary 2.4.** *Under the conditions of Theorem 2.3, if we choose* $\alpha = 2$*, then from (6) we obtain inequality for the exponentially* $(m_1,m_2)-$ *convex functions:*

$$\left|\frac{g(a_1m_1)+g(a_2m_2)}{2} - \frac{1}{m_2a_2-m_1a_1}\int_{a_1m_1}^{a_2m_2} g(x)dx\right| \leq \frac{(a_2m_2-a_1m_1)^2}{24}\left[\frac{m_1|g''(a_1)|}{e^{\lambda a_1}} + \frac{m_2|g''(a_2)|}{e^{\lambda a_2}}\right]. \tag{11}$$

**Corollary 2.5.** *Under the conditions of Theorem 2.3, if we choose $m_1 = m_2 = 1$ and $\lambda = 0$, then from (6) we obtain the inequality for convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}} \times U\right| \leq \frac{(a_2-a_1)^2}{2(\alpha+1)(\alpha+2)} \left[|g''(a_1)||+g''(a_2)|\right], \quad (12)$$

*where*

$$U = \frac{(\alpha+1)}{a_2-a_1}\left[J^{\alpha}_{a_1^+} g(a_2) + J^{\alpha}_{a_2^-} g(a_1)\right] - \left[J^{\alpha-1}_{a_1^+} g(a_2) + J^{\alpha-1}_{a_2^-} g(a_1)\right],$$

**Remark 2.6.** *In (12) if we choose $\alpha = 2$, then we obtain inequality for the convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2} - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \leq \frac{(a_2-a_1)^2}{24}\left[\left|g''(a_1)\right| + \left|g''(a_2)\right|\right].$$

*This inequality for convex functions obtained in [15](Proposition 2 ) and [13] (Corollary 3.1).*

**Corollary 2.7.** *Under the conditions of Theorem 2.3, if we choose $\lambda = 0$, then from (6) we obtain the inequality for $(m_1, m_2)-$convex functions:*

$$\begin{aligned} &\left|\frac{g(m_1a_1)+g(m_2a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}} \times U\right| \\ &\leq \frac{(m_2a_2-m_1a_1)^2}{2(\alpha+1)(\alpha+2)}\left[m_1\left|g''(a_1)\right| + m_2\left|g''(a_2)\right|\right], \end{aligned} \quad (13)$$

*where*

$$\begin{aligned} U =& \frac{(\alpha+1)}{m_2a_2-m_1a_1}\left[J^{\alpha}_{m_1a_1^+} g(m_2a_2) + J^{\alpha}_{m_2a_2^-} g(m_1a_1)\right] \\ &- \left[J^{\alpha-1}_{m_1a_1^+} g(m_2a_2) + J^{\alpha-1}_{m_2a_2^-} g(m_1a_1)\right]. \end{aligned}$$

**Remark 2.8.** *In (13) if we choose $m_1 = 1$, $m_2 = m$, then we obtain inequality for the $m-$convex functions:*

$$\begin{aligned} &\left|\frac{g(a_1)+g(ma_2)}{2} - \frac{\Gamma(\alpha+1)}{2(ma_2-a_1)^{\alpha-1}} \times U\right| \\ \leq\ & \frac{(ma_2-a_1)^2}{2(\alpha+1)(\alpha+2)}\left[\left|g''(a_1)\right| + m\left|g''(a_2)\right|\right], \end{aligned}$$

*where*

$$U = \frac{(\alpha+1)}{ma_2-a_1}\left[J^{\alpha}_{a_1^+} g(a_2) + J^{\alpha}_{a_2^-} g(a_1)\right] - \left[J^{\alpha-1}_{a_1^+} g(ma_2) + J^{\alpha-1}_{ma_2^-} g(a_1)\right],$$

**Corollary 2.9.** *Under the conditions of Theorem 2.3, if we choose $m_1 = 1$, $m_2 = m$, then from (6) we obtain inequality for the exponentially $m-$convex functions::*

$$\left|\frac{g(a_1)+g(ma_2)}{2} - \frac{\Gamma(\alpha+1)}{2(ma_2-a_1)^{\alpha-1}} \times U\right| \leq \frac{(ma_2-a_1)^2}{2(\alpha+1)(\alpha+2)} \times \left[\frac{|g''(a_1)|}{e^{\lambda a_1}} + \frac{m|g''(a_2)|}{e^{\lambda a_2}}\right], \tag{14}$$

*where*

$$U = \frac{(\alpha+1)}{ma_2-a_1}\left[J^{\alpha}_{a_1^+}g(ma_2) + J^{\alpha}_{ma_2^-}g(a_1)\right] - \left[J^{\alpha-1}_{a_1^+}g(ma_2) + J^{\alpha-1}_{ma_2^-}g(a_1)\right].$$

**Remark 2.10.** *In (14) if we choose $a_1 = 2, m = 1$, then we obtain the exponential type inequality. Some Exponential type inequality for first order derivative was obtained by Naila and Anwar (see in [14] Corollary (3.4):*

$$\left|\frac{g(a_1)+g(a_2)}{2} - \frac{1}{a_2-a_1}g(x)dx\right| \leq \frac{(a_2-a_1)^2}{24}\left[\frac{|g''(a_1)|}{e^{\lambda a_1}} + \frac{|g''(a_2)|}{e^{\lambda a_2}}\right].$$

**Theorem 2.11.** *Let $g : I = [0, a_2^*] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^\circ$. If $0 \leq a_1 < a_2 < a_2^*$, $g'' \in L[a_1m_1.a_1m_2]$, $g''$ positively defined and $|g''|^q$ is a exponentially $(m_1, m_2)-$ convex function, with $\xi \in (0,1)$, $\lambda \leq 0$, then for all $\alpha, q > 1$ the following inequality holds*

$$\left|\frac{g(a_1m_1)+g(a_2m_2)}{2} - \frac{\Gamma(\alpha+1)}{2(a_2m_2-a_1m_1)^{\alpha-1}} \times U\right| \leq \frac{(a_2m_2-a_1m_1)^2}{2} \times 2^{\frac{-1}{p}} \times H, \tag{15}$$

*where*

$$U = \frac{(\alpha+1)}{m_2a_2-a_1m_1}\left[J^{\alpha}_{a_1m_1^+}g(a_2m_2) + J^{\alpha}_{a_2m_2^-}g(a_1m_1)\right] - \left[J^{\alpha-1}_{a_1m_1^+}g(a_2m_2) + J^{\alpha-1}_{a_2m_2^-}g(a_1m_1)\right],$$

$$H = \mu \times \left\{\left[\frac{2m_1|g''(a_1)|^q}{(\alpha q+1)e^{\lambda a_1}} + \frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}} + \left[\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}} + \frac{2m_2|g''(a_2)|^q}{(\alpha q+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\},$$

$$\mu = [(\alpha q+2)(\alpha q+3)]^{-\frac{1}{q}}.$$

*Proof.* From Lemma 2.1 and from triangular inequality we obtain:

$$\left|\frac{g(a_1)+g(a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}} \times U\right| \leq \frac{(a_2m_2-a_1m_1)^2}{2}(|I_1|+|I_2|). \tag{16}$$

And using the well know Hölder integral inequality and since $|g''|^q$ is a exponentially $(m_1,m_2)$ convex function we can write as

$$\begin{aligned}|I_1| &= \left|\int_0^1 \xi(1-\xi)^\alpha g''(m_1a_1\xi+m_2(1-\xi)a_2 d\xi\right|\\ &\leq \int_0^1 \xi^{\frac{1}{p}}\xi^{\frac{1}{q}}(1-\xi)^\alpha |g''(m_1a_1\xi+m_2(1-\xi)a_2|d\xi\\ &\leq \left(\int_0^1 \xi^{\frac{1}{p}p}d\xi\right)^{\frac{1}{p}}\left[\frac{m_1}{e^{\lambda a_1}}|g''(a_1)|^q\int_0^1 \xi^2(1-\xi)^{\alpha q}d\xi\right.\\ &\qquad\left.+\frac{m_2}{e^{\lambda a_2}}|g''(a_2)|^q\int_0^1 \xi(1-\xi)(1-\xi)^{\alpha q}d\xi\right]^{\frac{1}{q}}\end{aligned}$$

or

$$\begin{aligned}|I_1| &\leq \frac{1}{\sqrt[p]{2}}\left[\frac{2m_1|g''(a_1)|^q}{(\alpha q+1)(\alpha q+2)(\alpha q+3)e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{(\alpha q+2)(\alpha q+3)e^{\lambda a_2}}\right]^{\frac{1}{q}} \qquad (17)\\ &= \frac{1}{\sqrt[p]{2}}\left(\frac{1}{(\alpha q+2)(\alpha q+3)}\right)^{\frac{1}{q}}\left[\frac{2m_1|g''(a_1)|^q}{(\alpha q+1)e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}.\end{aligned}$$

Similarly from $|I_2|$ we get the inequalities

$$|I_2| \leq \frac{1}{\sqrt[p]{2}}\left(\frac{1}{(\alpha q+2)(\alpha q+3)}\right)^{\frac{1}{q}}\left[\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2m_2|g''(a_2)|^q}{(\alpha q+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}. \qquad (18)$$

Adding inequalities (17) and (18) we get

$$|I_1|+|I_2| \leq 2^{\frac{-1}{p}}\times H.$$

Put this value in (16) inequality we get final result. □

**Corollary 2.12.** *Under the condition of Theorem 2.11, if we choose $\lambda=0$, then we obtained inequality for the $(m_1,m_2)-$convex functions:*

$$\left|\frac{g(a_1m_1)+g(a_2m_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m_2-a_1m_1)^{\alpha-1}}\times U\right| \leq \frac{(a_2m_2-a_1m_1)^2}{2\sqrt[p]{2}}\times H, \qquad (19)$$

*where*

$$U=\frac{(\alpha+1)}{m_2a_2-a_1m_1}\left[J^{\alpha}_{a_1m_1^+}g(a_2m_2)+J^{\alpha}_{a_2m_2^-}g(a_1m_1)\right]$$
$$-\left[J^{\alpha-1}_{a_1m_1^+}g(a_2m_2)+J^{\alpha-1}_{a_2m_2^-}g(a_1m_1)\right],$$
$$H=\mu\times\left\{\left[\frac{2m_1|g''(a_1)|^q}{\alpha q+1}+m_2\left|g''(a_2)\right|^q\right]^{\frac{1}{q}}+\left[m_1\left|g''(a_1)\right|^q+\frac{2m_2|g''(a_2)|^q}{\alpha q+1}\right]^{\frac{1}{q}}\right\},$$
$$\mu=[(\alpha q+2)(\alpha q+3)]^{-\frac{1}{q}}.$$

**Remark 2.13.** *In (19) if we choose $m_1=1$ and $m_2=m$, then we obtain inequality for the $m-$convex functions was proved in [13]*

$$\left|\frac{g(a_1)+g(a_2m)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m-a_1)^{\alpha-1}}\times U\right|\le\frac{(a_2m-a_1)^2}{2}\times 2^{\frac{-1}{p}}\times H, \tag{20}$$

*where*

$$U=\frac{(\alpha+1)}{ma_2-a_1}\left[J^{\alpha}_{a_1^+}g(a_2m)+J^{\alpha}_{a_2m^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2m)+J^{\alpha-1}_{a_2m^-}g(a_1)\right],$$
$$H=\mu\times\left\{\left[\frac{2|g''(a_1)|^q}{\alpha q+1}+m\left|g''(a_2)\right|^q\right]^{\frac{1}{q}}+\left[\left|g''(a_1)\right|^q+\frac{2m|g''(a_2)|^q}{\alpha q+1}\right]^{\frac{1}{q}}\right\},$$
$$\mu=[(\alpha q+2)(\alpha q+3)]^{-\frac{1}{q}}.$$

**Remark 2.14.** *In (20) if we choose $m=1$, then we obtained inequality for the convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}}\times U\right|\le\frac{(a_2-a_1)^2}{2}\times 2^{\frac{-1}{p}}\times H, \tag{21}$$

*where*

$$U=\frac{(\alpha+1)}{a_2-a_1}\left[J^{\alpha}_{a_1^+}g(a_2)+J^{\alpha}_{a_2^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2)+J^{\alpha-1}_{a_2^-}g(a_1)\right],$$
$$H=\mu\times\left\{\left[\frac{2|g''(a_1)|^q}{\alpha q+1}+\left|g''(a_1)\right|^q\right]^{\frac{1}{q}}+\left[\left|g''(a_1)\right|^q+\frac{2|g''(a_2)|^q}{\alpha q+1}\right]^{\frac{1}{q}}\right\},$$
$$\mu=[(\alpha q+2)(\alpha q+3)]^{-\frac{1}{q}}.$$

**Corollary 2.15.** *Under the condition of Theorem 2.11, if we choose $m_1=1$ ,$m_2=m$, then we obtained for the exponentially $m-$convex functions:*

$$\left|\frac{g(a_1)+g(a_2m)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m-a_1)^{\alpha-1}}\times U\right|\le\frac{(a_2m-a_1)^2}{2}\times 2^{\frac{-1}{p}}\times H, \tag{22}$$

*where*

$$U=\frac{(\alpha+1)}{a_2m-a_1}\left[J^{\alpha}_{a_1^+}g(a_2m)+J^{\alpha}_{a_2m^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2m)+J^{\alpha-1}_{a_2m^-}g(a_1)\right],$$

$$H=\mu\times\left\{\left[\frac{2|g''(a_1)|^q}{(\alpha q+1)e^{\lambda a_1}}+\frac{m|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}+\left[\frac{|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2m|g''(a_2)|^q}{(\alpha q+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\},$$

$$\mu=[(\alpha q+2)(\alpha q+3)]^{-\frac{1}{q}}.$$

**Remark 2.16.** *In (22) if we choose $m=1$, then we obtained inequality for the exponentially convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}}\times U\right|\le\frac{(a_2-a_1)^2}{2}\times 2^{\frac{-1}{p}}\times H, \tag{23}$$

*where*

$$U=\frac{(\alpha+1)}{a_2-a_1}\left[J^{\alpha}_{a_1^+}g(a_2)+J^{\alpha}_{a_2^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2)+J^{\alpha-1}_{a_2^-}g(a_1)\right],$$

$$H=\mu\times\left\{\left[\frac{2|g''(a_1)|^q}{(\alpha q+1)e^{\lambda a_1}}+\frac{|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}+\left[\frac{|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2|g''(a_2)|^q}{(\alpha q+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\},$$

$$\mu=[(\alpha q+2)(\alpha q+3)]^{-\frac{1}{q}}.$$

**Theorem 2.17.** *Let $g:I=[0,a_2^*]\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on $I^\circ$. If $0\le a_1<a_2<a_2^*$, $g''\in L[a_1m_1.a_1m_2]$, $g''$ positively defined and $|g''|^q$is a exponentially $(m_1,m_2)-$ convex function, with $\xi\in(0,1)$, $\lambda\le 0$, then for all $q\ge 1$, $\alpha>1$ the following inequality holds*

$$\left|\frac{g(a_1m_1)+g(a_2m_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m_2-a_1m_1)^{\alpha-1}}\times U\right|\le\frac{(a_2m_2-a_1m_1)^2}{2}\times\nu\times R, \tag{24}$$

*where*

$$U=\frac{(\alpha+1)}{m_2a_2-a_1m_1}\left[J^{\alpha}_{a_1m_1^+}g(a_2m_2)+J^{\alpha}_{a_2m_2^-}g(a_1m_1)\right]-\left[J^{\alpha-1}_{a_1m_1^+}g(a_2m_2)+J^{\alpha-1}_{a_2m_2^-}g(a_1m_1)\right],$$

$$\nu=[(\alpha+1)(\alpha+2)]^{\frac{1}{q}-1}$$

$$R=\omega\times\left\{\left[\frac{2m_1|g''(a_1)|^q}{(\alpha+1)e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}+\left[\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2m_2|g''(a_2)|^q}{(\alpha+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\},$$

$$\omega=[(\alpha+2)(\alpha+3)]^{-\frac{1}{q}}.$$

*Proof.* From Lemma 2.1 and from triangular inequality we obtained:

$$\left|\frac{g(a_1)+g(a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}}\times U\right|\leq\frac{(a_2m_2-a_1m_1)^2}{2}(|I_1|+|I_2|),\tag{25}$$

And using the well know power-mean integral inequality and since $|g''|^q$ is a exponentially $(m_1,m_2)$ convex function we can write as

$$\begin{aligned}|I_1| &= \left|\int_0^1 \xi(1-\xi)^\alpha g''(m_1a_1\xi+m_2(1-\xi)a_2 d\xi\right| \\ &\leq \left(\int_0^1 \xi(1-\xi)^\alpha d\xi\right)^{1-\frac{1}{q}}\left[\frac{m_1}{e^{\lambda a_1}}|g''(a_1)|^q\int_0^1 \xi^2(1-\xi)^\alpha d\xi\right. \\ &\qquad \left.+\frac{m_2}{e^{\lambda a_2}}|g''(a_2)|^q\int_0^1 \xi(1-\xi^{\alpha+1})d\xi\right]^{\frac{1}{q}} \\ &= \nu\times\omega\left[\frac{2m_1|g''(a_1)|^q}{(\alpha+1)e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}},\end{aligned}\tag{26}$$

Similarly, we can get for $|I_2|$ :

$$|I_2|\leq\nu\times\omega\left[\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2m_2|g''(a_2)|^q}{(\alpha+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}.\tag{27}$$

Adding inequalities (26) and (27), we get

$$|I_1|+|I_2|\leq\nu\times R.$$

Put this value in (25) inequality we get final result. □

**Corollary 2.18.** *Under the conditions of Theorem 2.17, if we choose $a_1=2, m_1=m_2=1$ and $\lambda=0$, then from (24) we obtain inequality for the convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2}-\frac{1}{a_2-a_1}g(x)dx\right|\leq\frac{(a_2-a_1)^2}{24}\times R,$$

*where*

$$R=\left[\frac{2|g''(a_1)|^q+3|g''(a_2)|^q}{5}\right]^{\frac{1}{q}}+\left[\frac{3|g''(a_1)|^q+2|g''(a_2)|^q}{5}\right]^{\frac{1}{q}}.$$

*This inequality for convex functions obtained in [15] (Proposition 6 ) and [13](Corollary 3.2).*

**Corollary 2.19.** *Under the condition of Theorem 2.17, if we choose and $\lambda=0$, then we obtained inequality for the $(m_1,m_2)-$convex functions:*

$$\left|\frac{g(a_1m_1)+g(a_2m_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m_2-a_1m_1)^{\alpha-1}}\times U\right|\leq\frac{(a_2m_2-a_1m_1)^2}{2}\times\nu\times R,\tag{28}$$

*where*

$$U = \frac{(\alpha+1)}{m_2a_2-a_1m_1}\left[J^{\alpha}_{a_1m_1^+}g(a_2m_2)+J^{\alpha}_{a_2m_2^-}g(a_1m_1)\right] - \left[J^{\alpha-1}_{a_1m_1^+}g(a_2m_2)+J^{\alpha-1}_{a_2m_2^-}g(a_1m_1)\right],$$

$$R = \omega\times\left\{\left[\frac{2m_1|g''(a_1)|^q}{\alpha+1}+m_2|g''(a_2)|^q\right]^{\frac{1}{q}}+\left[m_1|g''(a_1)|^q+\frac{2m_2|g''(a_2)|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\},$$

$$\nu = [(\alpha+1)(\alpha+2)]^{\frac{1}{q}-1}, \omega = [(\alpha+2)(\alpha+3)]^{-\frac{1}{q}}.$$

**Corollary 2.20.** *Under the condition of Theorem 2.17, if we choose $m_1 = 1, m_2 = m$, then we obtained inequality for the exponentially $m-$convex functions:*

$$\left|\frac{g(a_1)+g(a_2m)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m-a_1)^{\alpha-1}}\times U\right| \le \frac{(a_2m-a_1)^2}{2}\times\nu\times R, \tag{29}$$

*where*

$$U = \frac{(\alpha+1)}{a_2m-a_1}\left[J^{\alpha}_{a_1^+}g(a_2m)+J^{\alpha}_{a_2m^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2m)+J^{\alpha-1}_{a_2m^-}g(a_1)\right],$$

$$R = \omega\times\left\{\left[\frac{2|g''(a_1)|^q}{(\alpha+1)e^{\lambda a_1}}+\frac{m|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}+\left[\frac{|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2m|g''(a_2)|^q}{(\alpha+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\},$$

$$\nu = [(\alpha+1)(\alpha+2)]^{\frac{1}{q}-1}, \omega = [(\alpha+2)(\alpha+3)]^{-\frac{1}{q}}.$$

**Remark 2.21.** *In (29) if we choose $m = 1$, then we obtained inequality for the exponentially convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}}\times U\right| \le \frac{(a_2-a_1)^2}{2}\times\nu\times R, \tag{30}$$

*where*

$$U = \frac{(\alpha+1)}{a_2-a_1}\left[J^{\alpha}_{a_1^+}g(a_2)+J^{\alpha}_{a_2^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2)+J^{\alpha-1}_{a_2^-}g(a_1)\right],$$

$$R = \omega\times\left\{\left[\frac{2|g''(a_1)|^q}{(\alpha+1)e^{\lambda a_1}}+\frac{|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}+\left[\frac{|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2|g''(a_2)|^q}{(\alpha+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\},$$

$$\nu = [(\alpha+1)(\alpha+2)]^{\frac{1}{q}-1}, \omega = [(\alpha+2)(\alpha+3)]^{-\frac{1}{q}}.$$

**Remark 2.22.** *In (29) if we choose $\lambda = 0$, then we obtained for the $m-$convex functions:*

$$\left|\frac{g(a_1)+g(a_2m)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2m-a_1)^{\alpha-1}}\times U\right| \le \frac{(a_2m-a_1)^2}{2}\times\nu\times R, \tag{31}$$

*where*

$$U=\frac{\alpha+1}{ma_2-a_1}\left[J^{\alpha}_{a_1^+}g(a_2m)+J^{\alpha}_{a_2m^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2m)+J^{\alpha-1}_{a_2m^-}g(a_1)\right],$$
$$R=\omega\times\left\{\left[\frac{2|g''(a_1)|^q}{\alpha+1}+m\left|g''(a_2)\right|^q\right]^{\frac{1}{q}}+\left[\left|g''(a_1)\right|^q+\frac{2m|g''(a_2)|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\},$$
$$\omega=[(\alpha+2)(\alpha+3)]^{-\frac{1}{q}},\nu=[(\alpha+1)(\alpha+2)]^{\frac{1}{q}-1}.$$

**Remark 2.23.** *In (31) if we choose $m=1$, then we obtained for the convex functions:*

$$\left|\frac{g(a_1)+g(a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha-1}}\times U\right|\leq\frac{(a_2-a_1)^2}{2}\times\nu\times R, \tag{32}$$

*where*

$$U=\frac{(\alpha+1)}{a_2-a_1}\left[J^{\alpha}_{a_1 1^+}g(a_2)+J^{\alpha}_{a_2^-}g(a_1)\right]-\left[J^{\alpha-1}_{a_1^+}g(a_2)+J^{\alpha-1}_{a_2^-}g(a_1)\right],$$
$$R=\omega\times\left\{\left[\frac{2|g''(a_1)|^q}{\alpha+1}+\left|g''(a_2)\right|^q\right]^{\frac{1}{q}}+\left[\left|g''(a_1)\right|^q+\frac{2|g''(a_2)|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\},$$
$$\omega=[(\alpha+2)(\alpha+3)]^{-\frac{1}{q}},\nu=[(\alpha+1)(\alpha+2)]^{\frac{1}{q}-1}.$$

**Theorem 2.24.** *Let $g:I=[0,a_2^*]\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on $I^\circ$. If $0\leq a_1<a_2<a_2^*$, $g''\in L[0,a_2^*]$, $g''$ positively defined and $|g''|^q$ is a exponentialy $(m_1,m_2)-$convex function, whith $m_1,m_2\in(0,1]$, $\lambda\leq 0$, $\frac{1}{p}+\frac{1}{q}=1$, then for all $\alpha>1$ the following inequality holds*

$$\begin{aligned}&\left|\frac{g(m_1a_1)+g(m_2a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}}\times U\right| \\ \leq\;&\frac{m_2a_2-m_1a_1}{2\cdot 3^{\frac{1}{q}}}\left[a_2^{\frac{1}{p}}(p+1,\alpha p+2)+a_2^{\frac{1}{p}}(p+2,\alpha p+1)\right]\cdot Q(m_1,m_2,\lambda),\end{aligned} \tag{33}$$

*where*

$$U = \frac{(\alpha+1)}{m_2a_2-m_1a_1}\left[J^{\alpha}_{m_1a_1^+}g(m_2a_2)+J^{\alpha}_{m_2a_2^-}g(m_1a_1)\right] - \left[J^{\alpha-1}_{m_1a_1^+}g(m_2a_2)+J^{\alpha-1}_{m_2a_2^-}g(m_1a_1)\right],$$

$$Q(m_1,m_2,\lambda) = \left(\frac{m_1\left|g''(a_1)\right|^q}{2e^{\lambda a_1}}+\frac{m_2\left|g''(a_2)\right|^q}{e^{\lambda a_2}}\right)^{\frac{1}{q}} + \left(\frac{m_1\left|g''(a_1)\right|^q}{e^{\lambda a_1}}+\frac{m_2\left|g''(a_2)\right|^q}{2e^{\lambda a_2}}\right)^{\frac{1}{q}},$$

$a_2(.,.)-$ *Euler Beta function.*

*Proof.* From Lemma 2.1 and the properties of the module we have:

$$\left|\frac{g(m_1a_1)+g(m_2a_2)}{2}-\frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}}\times U\right| \le \frac{(m_2a_2-m_1a_1)^2}{2}\left(|I_1|+|I_2|\right). \quad (34)$$

Considering the fact that $|g''|^q$ is an exponentially $(m_1,m_2)-$convex function and from Hölder-Işcan inequalities Theorem 1.2 for the first integral we get:

$$\begin{aligned} |I_1| &\le \int_0^1 \xi(1-\xi)^{\alpha}\left|g''(m_1a_1\xi+(1-\xi)m_2a_2)\right|d\xi \quad (35)\\ &\le \frac{1}{m_2a_2-m_1a_1}\left\{\left(\int_0^1 \xi^p(1-\xi)^{\alpha p+1}d\xi\right)^{\frac{1}{p}}\right.\\ &\quad\times\left(\int_0^1(1-\xi)\left|g''(m_1a_1\xi+(1-\xi)m_2a_2)\right|^q d\xi\right)^{\frac{1}{q}}\\ &\left.+\left(\int_0^1 \xi^{p+1}(1-\xi)^{\alpha p}d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \xi\left|g''(m_1a_1\xi+(1-\xi)m_2a_2)\right|^q d\xi\right)^{\frac{1}{q}}\right\}\\ &\le \frac{1}{m_2a_2-m_1a_1}\left\{a_2^{\frac{1}{p}}(p+1,\alpha p+2)\left(\frac{m_1\left|g''(a_1)\right|^q}{6e^{\lambda a_1}}+\frac{m_2\left|g''(a_2)\right|^q}{3e^{\lambda a_2}}\right)^{\frac{1}{q}}\right.\\ &\left.+a_2^{\frac{1}{p}}(p+2,\alpha p+1)\left(\frac{m_1\left|g''(a_1)\right|^q}{3e^{\lambda a_1}}+\frac{m_2\left|g''(a_2)\right|^q}{6e^{\lambda a_2}}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Similarly, for the second integral we get:

$$\begin{aligned} |I_2| &\leq \int_0^1 \xi(1-\xi)^{\alpha} \left|g''(m_1a_1(1-\xi)+\xi m_2a_2)\right| d\xi \\ &\leq \frac{a_2^{\frac{1}{p}}(p+1,\alpha p+2)}{m_2a_2-m_1a_1}\left(\frac{m_1|g''(a_1)|^q}{3e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{6e^{\lambda a_2}}\right)^{\frac{1}{q}} \\ &+ \frac{a_2^{\frac{1}{p}}(p+2,\alpha p+1)}{m_2a_2-m_1a_1}\left(\frac{m_1|g''(a_1)|^q}{6e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{3e^{\lambda a_2}}\right)^{\frac{1}{q}}. \end{aligned} \tag{36}$$

Adding inequalities (35) and (36) we get:

$$\begin{aligned} |I_1|+|I_2| \leq & \frac{3^{-\frac{1}{q}}}{m_2a_2-m_1a_1}\left[a_2^{\frac{1}{p}}(p+1,\alpha p+2)+a_2^{\frac{1}{p}}(p+2,\alpha p+1)\right] \\ & \times\left[\left(\frac{m_1|g''(a_1)|^q}{2e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{2e^{\lambda a_2}}\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{37}$$

Inequality (33) follows from (34) and (37). Thus, proof is completely done. □

**Theorem 2.25.** *Let $g : I = [0, a_2^*] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^{\circ}$. If $0 \leq a_1 < a_2 < a_2^*$, $g'' \in L[0, a_2^*]$, $g''$ positively defined and $|g''|^q$ is a exponentialy $(m_1, m_2)-$ convex function, with $m_1, m_2 \in (0,1]$, $\lambda \leq 0$, $\frac{1}{p}+\frac{1}{q}=1$, then for all $\alpha > 1$ the following inequality holds*

$$\begin{aligned} & \left|\frac{g(m_1a_1)+g(m_2a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}} \times U\right| \\ \leq & \frac{m_2a_2-m_1a_1}{2(\alpha+2)(\alpha+3)}\left[\Phi(\alpha,q)\cdot P(m_1,m_2,\lambda)+\Psi(\alpha,q)\cdot R(m_1,m_2,\lambda)\right], \end{aligned} \tag{38}$$

*where*

$$U = \frac{(\alpha+1)}{m_2a_2 - m_1a_1}\left[J^{\alpha}_{m_1a_1^+}g(m_2a_2) + J^{\alpha}_{m_2a_2^-}g(m_1a_1)\right] - \left[J^{\alpha-1}_{m_1a_1^+}g(m_2a_2) + J^{\alpha-1}_{m_2a_2^-}g(m_1a_1)\right],$$

$$\Phi(\alpha,q) = \left(\frac{\alpha+2}{\alpha+4}\right)^{\frac{1}{q}}, \Psi(\alpha,q) = \frac{2^{\frac{1}{q}}}{\alpha+1}\left(\frac{\alpha+1}{\alpha+4}\right)^{\frac{1}{q}},$$

$$P(m_1,m_2,\lambda) = \left[\frac{2m_1\left|g''(a_1)\right|^q}{(\alpha+4)e^{\lambda a_1}} + \frac{m_2\left|g''(a_2)\right|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}} + \left[\frac{m_1\left|g''(a_1)\right|^q}{e^{\lambda a_1}} + \frac{2m_2\left|g''(a_2)\right|^q}{(\alpha+4)\ e^{\lambda a_2}}\right]^{\frac{1}{q}},$$

$$R(m_1,m_2,\lambda) = \left[\frac{3m_1\left|g''(a_1)\right|^q}{(\alpha+1)\ e^{\lambda a_1}} + \frac{m_2\left|g''(a_2)\right|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}} + \left[\frac{m_1\left|g''(a_1)\right|^q}{e^{\lambda a_1}} + \frac{3m_2\left|g''(a_2)\right|^q}{(\alpha+1)e^{\lambda a_2}}\right]^{\frac{1}{q}}.$$

*Proof.* From Lemma 2.1 and the properties of the module we have:

$$\left|\frac{g(m_1a_1)+g(m_2a_2)}{2} - \frac{\Gamma(\alpha+1)}{2(m_2a_2-m_1a_1)^{\alpha-1}} \times U\right| \leq \frac{(m_2a_2-m_1a_1)^2}{2}\left(|I_1|+|I_2|\right). \quad (39)$$

Considering the fact that $|g''|^q$ is an exponentially $(m_1,m_2)-$convex function and from Improved Power mean inequalities Theorem 1.3 for the first integral we get:

$$
\begin{aligned}
|I_1| &\le \int_0^1 \xi(1-\xi)^{\alpha}\left|g''(m_1a_1\xi+(1-\xi)m_2a_2)\right|d\xi \le \frac{1}{m_2a_2-m_1a_1}\\
&\times\left\{\left(\int_0^1 \xi(1-\xi)^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 \xi(1-\xi)^{\alpha+1}\left|g''(m_1a_1\xi+(1-\xi)m_2a_2)\right|^q d\xi\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \xi^2(1-\xi)^{\alpha}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 \xi^2(1-\xi)^{\alpha}\left|g''(m_1a_1\xi+(1-\xi)m_2a_2)\right|^q d\xi\right)^{\frac{1}{q}}\right\}\\
&\le \frac{1}{m_2a_2-m_1a_1}\left(\frac{1}{(\alpha+2)(\alpha+3)}\right)^{1-\frac{1}{q}}\left(\frac{2m_1|g''(a_1)|^q}{(\alpha+2)(\alpha+3)(\alpha+4)e^{\lambda a_1}}\right.\\
&\qquad\left.+\frac{m_2|g''(a_2)|^q}{(\alpha+3)(\alpha+4)e^{\lambda a_2}}\right)^{\frac{1}{q}}\\
&+\frac{1}{m_2a_2-m_1a_1}\left(\frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{1-\frac{1}{q}}\left(\frac{6m_1|g''(a_1)|^q e^{-\lambda a_1}}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\right.\\
&\qquad\left.+\frac{2m_2|g''(a_2)|^q e^{-\lambda a_2}}{(\alpha+2)(\alpha+3)(\alpha+4)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Or

$$
\begin{aligned}
|I_1| \le \frac{1}{(\alpha+2)(\alpha+3)}&\left\{\left(\frac{\alpha+2}{\alpha+4}\right)^{\frac{1}{q}}\left[\frac{2m_1|g''(a_1)|^q}{(\alpha+4)e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{2^{\frac{1}{q}}}{(\alpha+1)}\left(\frac{\alpha+1}{\alpha+4}\right)^{\frac{1}{q}}\left[\frac{3m_1|g''(a_1)|^q}{(\alpha+1)\,e^{\lambda a_1}}+\frac{m_2|g''(a_2)|^q}{e^{\lambda a_2}}\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{40}
$$

Similarly, for the second integral we get:

$$
\begin{aligned}
|I_2| \le \frac{(m_2a_2-m_1a_1)^{-1}}{(\alpha+2)(\alpha+3)}&\left\{\left(\frac{\alpha+2}{\alpha+4}\right)^{\frac{1}{q}}\left[\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{2m_2|g''(a_2)|^q}{(\alpha+4)\,e^{\lambda a_2}}\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{2^{\frac{1}{q}}}{(\alpha+1)}\left(\frac{\alpha+1}{\alpha+4}\right)^{\frac{1}{q}}\left[\frac{m_1|g''(a_1)|^q}{e^{\lambda a_1}}+\frac{3m_2|g''(a_2)|^q}{(\alpha+1)\,e^{\lambda a_2}}\right]\right\}^{\frac{1}{q}}.
\end{aligned} \tag{41}
$$

Adding inequalities (39) and (41) and taking into account the accepted designations, we get:

$$
|I_1|+|I_2| \le \Phi(\alpha,q)\cdot P(m_1,m_2,\lambda)+\Psi(\alpha,q)\cdot R(m_1,m_2,\lambda). \tag{42}
$$

Inequality (38) follows from inequalities (39) and (42). Thus, proof is completely done. □

### Competing interests

The authors declare that they have no competing interests.


### Acknowledgements

The authors are grateful to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.


### Author's contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

### References


[1] Butt, S. I., Nadeem, M., Qaisar, S., Akdemir, A. O. Abdeljawad,T. (2020). Hermite-Jensen-Mercer type inequalities for conformable integrals and related results, *Adv. Differ. Equ.* **1**, 1-24.

[2] Khan, S., Khan, M. A., Butt, S. I., Chu, Y. M. (2020). A new bound for the Jensen gap pertaining twice differentiable functions with applications, *Adv. Differ. Equ.* **1**, 1-11.

[3] Mehmood, N., Butt, S. I., Pečarić, D., Pečarić, J. (2020). Generalizations of cyclic refinements of Jensen inequality by Lidstone polynomial with applications in Information Theory, *Journal of Mathematical Inequalities*. **14** (1), 249-271.

[4] Bayraktar, B. (2020). Some Integral Inequalities For Functions Whose Absolute Values Of The Third Derıvative is Concave And $r-$Convex, *Turkish J. Ineq.*, **4** (2), 59 –78 .

[5] Bayraktar, B. (2020). Some New Generalizations Of Hadamard–Type Midpoint Inequalities Involving Fractional Integrals, *Probl. Anal. Issues Anal.* **9** (27-3), 66–82, DOI: 10.15393/j3.art.2020.8270

[6] Bayraktar, B. (2020). Some New Inequalities of Hermite-Hadamard Type for Differentiable Godunova-Levin Functions via Fractional Integrals, *Konuralp Journal of Mathematics*, **8**(1), 91–96.

[7] Özdemir, M. E. Butt, S. I. Bayraktar B., Nasir, J. (2020). Several integral inequalities for $(\alpha, s, m)-$convex functions, *AIMS Mathematics*, **5**(4), 3906–3921. DOI: 10.3934/math.2020253.

[8] Kadakal, H. (2019). $(\alpha, m_1, m_2)-$ convexity and some inequalities of Hermite-Hadamard type, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **68** (2), 2128-2142, DOI: 10.31801/cfsuasmas.511184

[9] Awan, M. U., Noor, M. A., Noor, K. I. (2018) Hermite–Hadamard inequalities for exponentially convex functions. *Appl. Math. Inf. Sci.* **12**(2), 405–409 .

[10] Butt, S. I., Bayraktar, B., Nasir, J. (2021). Some Generalized Ostrowski type inequalities via exponentially $(m_1, m_2)-$convex functions with related applications (Sabmitted).

[11] Işcan, I. (2019). New refinements for integral and sum forms of Hölder inequality, *Journal of Inequalities and Applications*, **2019**:30.

[12] Kadakal, M., Işcan, I., Kadakal H., Bekar, K. (2019) On improvements of some integral inequalities, Researchgate, DOI: 10.13140/RG.2.2.15052.46724, Preprint, January 2019.

[13] Bayraktar, B. (2020). Some Integral Inequalities Of Hermite-Hadamard Type For Differentiable $(s, m)-$Convex Functions Via Fractional Integrals. *TWMS Journal of Applied and Engineering Mathematics*, **10**(3), 625–637.

[14] Naila, M., Anwar, M. (2019). Hermite-Hadamard type inequalities for exponentially $p-$ convex functions and exponentially $s-$convex functions in the second sense with applications. *Journal of Inequalities and Applications*, **1** 2019: 92.

[15] Sarıkaya, M. Z., Aktan, N. (2011). On the generalization of some integral inequalities and their applications. *Math. Comput. Modelling*, **54**, 2175–2182.