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DIFFERENCE SCHEME METHOD FOR THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this study, wave equations with initial boundary conditions have been studied. The general form of the wave equation has been derived. The first order and second order difference schemes were established for the presented IBVP. The stability of the difference schemes has been guaranteed. The approximation solution of the problem was achieved by using finite difference methods. Two different examples are provided. A comparison between the exact and approximation solution has been carried out. Absolute errors of the problem have been presented by using MATLAB software. Moreover, the comparison shows that the second order difference scheme is a more accurate result than the first order. It is shown that the results of the comparison guaranty the reliability and accuracy of the presented method.

Keywords: Wave equation, IBVP, Numerical Solution, Finite difference method.

1. INTRODUCTION

In the last decades, Partial differential equations (PDE) have focused on many studies because of their common appearance in several applications in mathematics, physics, seismology, science, finance, engineering, and mechanics [1, 2]. PDEs are important for modeling a wide variety of problems in science and engineering, including electrodynamics, elasticity, wave propagation, signal analysis, and thermodynamics [3–5].
However, the exact solution of some types of PDE is known [1–3]. Much attention has been given to studies to test the reliability and accuracy of the approximation and numerical techniques such as finite difference methods [4–6]. The wave equation is an important second order linear PDE for describing waves, such as mechanical waves or light waves [1–5]. It is the simplest example of the hyperbolic differential equation. Wave equations and modifications play fundamental roles in continuum mechanics, quantum mechanics, plasma physics, general relativity, geophysics, and many other scientific and technical disciplines [3–5]. Nytrebych et al. presented an analytical method for studying the wave

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process mathematical model under two-point time conditions[6].
Moreover, several different approximation methods were presented for solving wave equations [7]. Finite difference approximations have been used for solving different types of partial differential equations in the works [8–15]. Qian and Weiss worked on Wavelets and the numerical solution of partial differential equations [11]. Ashyralev and Modanli presented a finite difference method for finding a numerical solution for telegraph PDE [17]. Sweilam, et al. used the finite difference method to find numerical solutions of two-sided space-fractional wave equations [18]. Carcione and Helle presented a Numerical solution of the wave equation on a staggered mesh [19]. Saadatmandi and Dehghan worked on a onedimensional wave equation with an integral condition [20, 21]. On the other hand, various studies on stability estimates have been done for different types of PDEs [21–25]. However, different numerical methods are presented for solving wave equations with initial boundary conditions [19–22]. In this work, we will present a numerical method based on difference scheme method for solving wave equations.

2. Deriving the 1D wave equation

In this section, we will present the derivation of wave equations which have been done in [25]. The derivation of the one-dimensional wave equation starts from Newtons and Hookes law. As shown in Figure 1. the key notion is that the restoring force due to tension on the string will be proportional to the curvature at the point. Then the force, ku_{xx} should be equal to mass times acceleration ρu_{tt}, where $c = \sqrt{k/\rho}$ turns out to be the velocity of propagation [25], we obtain

$$u_{tt} = c^2 u_{xx}$$

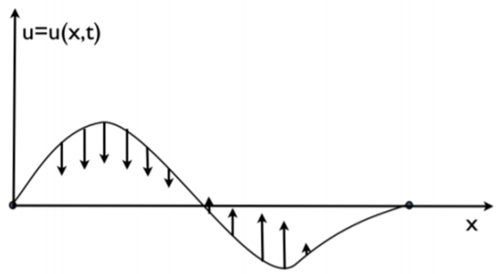

FIGURE 1. The restoring forces on a vibrating string, proportional to curvature.

Let $u = u(x,t)$ denote the displacement of a string from the neutral position $u \equiv 0$. Let $k = k(x)$, be the elasticity, and $\rho = \rho(x)$ be the mass density of the string. Especially, We do not assume the string is uniform in this derivation. Consider a small piece of string in the interval $[x, x+\Delta x]$. Its velocity be $u_t(x,t)$, it is mass $\rho(x)\Delta x$, and thus its kinetic energy is:

$$\Delta K = \frac{1}{2}\rho.(u_t)^2\Delta x.$$

The total kinetic energy of the string is provided by an integral,

$$K = \frac{1}{2}\int_0^L \rho.(u_t)^2 dx.$$

The potential energy for a string is $(\frac{k}{2})y^2$ given from Hookes law, where y is the length of the spring [25]. The length of the string is presented by arclength, in the case of a stretched string, as follow

$$ds = \sqrt{1+u_x^2}dx,$$

and potential energy is given in the following form

$$P = \int_0^L \frac{k}{2}(1+u_x^2)dx.$$

From it, the function u is presented as the integral over time of the difference of these two energies, so

$$L(u) = \frac{1}{2}\int_0^T\int_0^L \rho.(u_t)^2 - k.[1+(u_x)^2]dx\,dt.$$

Adding δ times a perturbation $h = h(x,t)$ to the function u, we obtain

$$L(u+\delta h) = L(u) + \delta\int_0^T\int_0^L \rho.u_t.h_t - k.u_x h_x dxdt + \text{highr order in } \delta.$$

The principle of least action says that for u to be a physical solution, the first-order term should vanish for any perturbation h. By applying integration by parts, we obtain

$$0 = \int_0^T\int_0^L (-\rho.u_{tt} + k.u_{xx} + k_x.u_x).hdxdt.$$

Because of all choices of h, this integral is zero, the first factor in the integral must be zero, then we get the wave equation for an inhomogeneous medium,

$$\rho.u_{tt} = k.u_{xx} + k_x.u_x$$

above equation reduces to the usual two-term wave equation, when the elasticity k is constant,

$$u_{tt} = c^2 u_{xx}$$

where the velocity $c = \sqrt{\frac{k}{\rho}}$, varies for changing density [25].

By applying initial boundary conditions, we can state the problem as follow:

$$\begin{cases} u_{tt}(t,x) = c^2 u_{xx}(t,x) & 0 \leq x \leq L, \\ u(t,0) = u(t,\pi) = 0, & 0 \leq t \leq T, \\ u(0,x) = \psi(x),\ u_t(0,x) = \varphi(x). \end{cases} \tag{1}$$

Where $\psi(x), \varphi(x)(x \in [0,\pi])$ and $f(t,x) = 0, ((t,x) \in [0,1] \times [0,\pi])$ are smooth functions, problem (1) presents a wave equation [11].

3. Stability and Difference scheme

This section introduces the explicit and implicit methods for approximate wave equation solutions in (1). Moreover, von Neumann's Stability Analysis will be presented. A Simple Explicit Method can be applied to (1), by using central differences for both time and space derivatives, we obtain

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} = c^2 \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \tag{2}$$

If we set $a = \dfrac{c\tau}{h}$, we can simplify (2), we get

$$u_n^{k+1} = -u_n^{k-1} + 2(1-a)u_n^k + a^2\left(u_{n+1}^k + u_{n-1}^k\right) \tag{3}$$

The stability of the explicit scheme in (3) guaranteed in [24–26], we state it with the usual ansatz [25],

$$\varepsilon_n^k = g^k e^{ikx_n}.$$

From which we obtain the following expression for the amplification factor $g(k)$

$$g^2 = 2(1-a^2)g - 1 + 2a^2 g\ cos(k\Delta x).$$

After several transformations, the last expression becomes just a quadratic equation for g, namely

$$g^2 - 2\beta g + 1 = 0, \tag{4}$$

where $\beta = 1 - 2a^2 sin^2\left(\dfrac{k\Delta x}{2}\right)$. By using quadratic rule solutions of the equation for $g(k)$ is

$$g_{1,2} = \beta \mp \sqrt{\beta^2 - 1}.$$

Notice that if $|\beta| > 1$ then at least one of the absolute values of $g_{1,2}$ is bigger than one. Therefore one should desire for $\beta| < 1$, we get:

$$g_{1,2} = \beta \mp i\sqrt{\beta^2 - 1}.$$

and $|g|^2 = \beta^2 + 1 - \beta^2 = 1$.
That is, scheme (3) is conditional stable. The stability condition read

$$-1 \leq 1 - 2a^2 sin^2(\frac{kh}{2}) \leq 1$$

what is equivalent to the standard CFL condition $a = \dfrac{c\tau}{h} \leq 1$.
For the wave equation in (1), we will introduce an implicit scheme from (2). However, the second-order difference scheme is used for presenting the conditional stability of the problem. By replacing all terms on the right-hand side of (2) by an average from the values to the time steps $k-1$ and $k+1$, we obtain

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} = \frac{c^2}{2h^2}\left(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}\right) \tag{5}$$

The difference equation in (5) is called a second order difference scheme for the problem (1). We used the implicit method. The stability of the difference scheme (5) is guaranteed in [25–28]. We will state it is stability, by using the standard ansatz

$$\varepsilon_n^k = g^k e^{ikx_n}.$$

Leading to the equation for $g(k)$

$$\beta g^2 - 2g + \beta = 0,$$

where $\beta = 1 + 2a^2 sin^2\left(\dfrac{kh}{2}\right)$. We have $\beta \geq 1$ for all k. Hence the solutions $g_{1,2}$ take the form

$$g_{1,2} = \frac{1 \mp i\sqrt{1-\beta^2}}{\beta},$$

and

$$|g|^2 = \frac{1-(1-\beta^2)}{\beta^2} = 1.$$

That is, the implicit scheme (5) is absolute stable.

4. The Method

Both difference equations in (2) and (5) can be rewritten to the following matrices formulas are obtained as

$$AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \qquad 1 \leq n \leq M-1, \quad u_0 = u_M = 0. \tag{6}$$

where, A, B and C are $(N+1)\times(N+1)$ matrix, U_{n+1}, U_n, U_{n-1} and φ_n is $(N+1)\times 1$ vectors, in the present work $\varphi_n = 0$ [11–13]. We applied a modified Gauss elimination method to solve the difference equation in (6). Then, we are looking for a matrix equation solution, which was presented as follows [16]:

$$u_j = \alpha_{j+1}u_{j+1} + \beta_{j+1}; \qquad u_M = 0;\ j = M-1, \ldots, 2, 1.$$

Where β_j are $(N+1)\times 1$ column vectors, and α_j are $(N+1)\times(N+1)$ square matrices, defined by

$$\alpha_{j+1} = -(B+C\alpha_j)^{-1}A,$$

and

$$\beta_{j+1} = (B+C\alpha_j)^{-1}(D\varphi - C\beta_j), \quad j = 1,2,\ldots,M-1,$$

where $j = 1,2,\ldots,M-1$, β_1 is the $(N+1)\times 1$ zero column vector, and α_1 is the $(N+1)\times(N+1)$ zero matrix. The Matlab program computes the results. For three distinct values of M and N, the comparison was made between numerical and exact solutions. The maximum error indicated where $h = \frac{\pi}{M}$, and $\tau = \frac{1}{N}$. Maximum absolute error calculated by

$$E_M^N = \max_{\substack{1\le k\le N-1\\ 1\le n\le M-1}} \left| u(t_k,x_n) - u_n^k \right|$$

where exact solution denoted by $u(t_k,x_n)$ and approximation solution denoted by u_n^k at points (t_k,x_n).

5. Numerical Computation

This section will apply the first order difference scheme in (2) and the second order difference scheme in (5) for two different numerical examples for the wave PDE with initial-boundary conditions. For both examples, obtained exact solutions will be compared with the numerical result for testing the method's ability. In applications, let us consider the general form of the IBVP hyperbolic wave partial differential equation

$$\begin{cases} u_{tt}(t,x) = c^2 u_{xx}(t,x) & 0\le x\le L, \\ u(t,0) = u(t,\pi) = 0, & 0\le t\le T, \\ u(0,x) = \psi(x),\ u_t(0,x) = \varphi(x). \end{cases}$$

Example 5.1. *Consider the following IBVP for the wave equation*

$$\begin{cases} u_{tt}(t,x) = u_{xx}(t,x) & 0\le x\le L, \\ u(t,0) = u(t,\pi) = 0, & 0\le t, \\ u(0,x) = sin(x),\ u_t(0,x) = 0. \end{cases} \tag{7}$$

Analytical solution of the problem (7) can be found by using one of the exact methods, which is $u(t,x) = cos(t)sin(x)$. Then the first order difference scheme for (7) is presented as follow:

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} = \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \\ u_0^k = u_\pi^k = 0, \qquad 0\le n\le N, 0\le m\le M \\ u_n^0 = sin(x_n),\ \dfrac{u_n^1 - u_n^0}{\tau} = 0 \end{cases} \tag{8}$$

For the second order difference scheme, we obtain

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} = \dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} \\ u_0^k = u_\pi^k = 0, \qquad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = sin(x_n), \ \dfrac{u_n^1-u_n^0}{\tau} = \dfrac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2} \end{cases} \tag{9}$$

The error analysis of the problem (7) is shown in table 1. However, the figures of exact and approximation solutions have been shown in figure 2.

Example 5.2. *Investigate the following IBVP for wave partial differential equation*

$$\begin{cases} u_{tt}(t,x) = u_{xx}(t,x) & 0 \le x \le L, \\ u(t,0) = u(t,\pi) = 0, & 0 \le t, \\ u(0,x) = 4sin(x) - sin(2x) - 3sin(5x), \ u_t(0,x) = 0. \end{cases} \tag{10}$$

The exact solution of the problem is $u(x,t) = 4cos(t)sin(x) - cos(2t)sin(2x) - 3cos(5t)sin(5x)$.
The first order difference scheme for the problem (10) is

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} = \dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} \\ u_0^k = u_\pi^k = 0, \qquad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = 4sin(x_n) - sin(2x_n) - 3sin(5x_n), \ \dfrac{u_n^1-u_n^0}{\tau} = 0 \end{cases} \tag{11}$$

The second order difference scheme (Implicit method) can be obtained as follow:

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} = \dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} \\ u_0^k = u_\pi^k = 0, \qquad 0 \le n \le N, \ 0 \le m \le M \\ u_n^0 = 4sin(x_n) - sin(2x_n) - 3sin(5x_n), \ \dfrac{u_n^1-u_n^0}{\tau} = \dfrac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2} \end{cases} \tag{12}$$

The error analyzes of problem (10) is shown in table 2. However, the figures of exact and approximation solutions have been shown in figure 3.

TABLE 1. Error analysis for exact and approximation solution of Example 5.1

$h=\dfrac{\pi}{M},\ \tau=\dfrac{1}{N}$	N=M=25	N=M=50	N=M=100
Difference scheme in (8)	1.727910^{-2}	8.537310^{-3}	4.238210^{-3}
Difference scheme in (9)	8.448210^{-4}	2.101510^{-4}	5.234110^{-5}

TABLE 2. Error analysis for exact and approximation solution of Example 5.2

$h=\dfrac{\pi}{M},\ \tau=\dfrac{1}{N}$	N=M=25	N=M=50	N=M=100
Difference scheme in (11)	5.564610^{-1}	2.354810^{-1}	1.044410^{-1}
Difference scheme in (12)	3.475710^{-1}	9.122210^{-2}	2.277110^{-2}

The exact and approximation solutions of given examples are presented in the following figures.

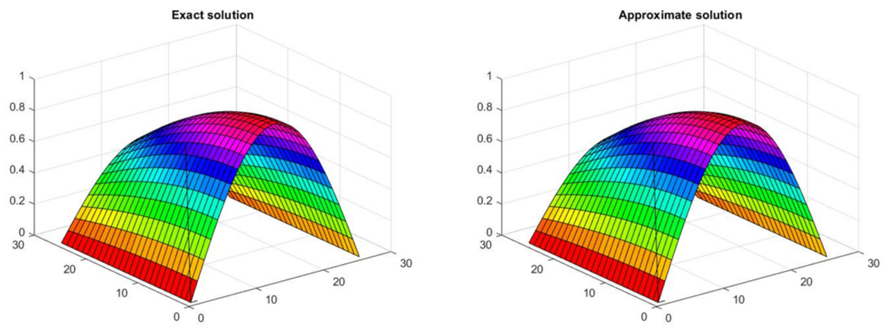

FIGURE 2. Exact solution and approximation solution of the problem (7), where N=M=25.

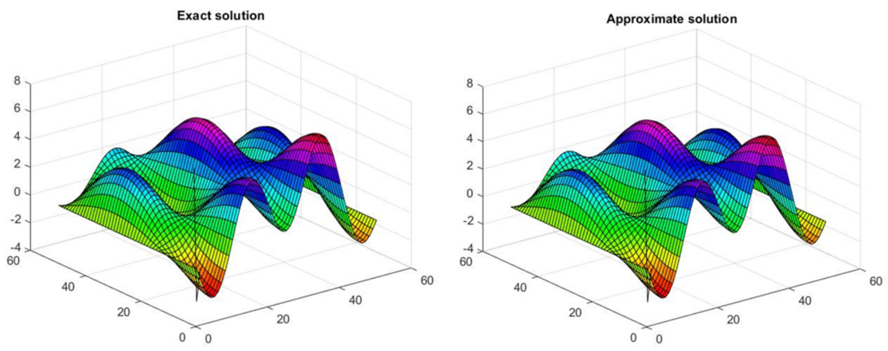

FIGURE 3. Exact solution and approximation solution of the problem (10), where N=M=50.

6. CONCLUSION

In the presented article, the wave equation was modeled mathematically, and the problem was verified. The initial boundary value problem for the wave equation has been discussed. The difference schemes of the first order and second order of accuracy for wave partial differential equations are showed. Stability for the problem and difference scheme has been established. Modified Gauss elimination technique based on difference equations used for achieving approximation results. An algorithm has been established, and the programs are written by MATLAB software for numerical computation. Two different examples with the analytical solution are given. Approximation results compared with the exact solution. Good results were obtained for the problem. Error analysis tables have been presented, and it is guaranteed the accuracy of the method. The comparison shows that the second order difference scheme is a more accurate result than the first order. It is worth mentioning that the technique can be used as a very accurate algorithm for the presented type of wave PDE. The accuracy of the numerical results supports theoretical statements of the presented solution of finite difference methods.

COMPETING INTERESTS

The authors declare that they have no competing interests.

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AUTHOR'S CONTRIBUTIONS

All authors equally contributed to this work. All authors read and approved the final manuscript.

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