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TOPOLOGIES GENERATED BY RAW BINARY STRUCTURES

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Abstract. We define three topologies induced by a raw binary operation which is weaker than the concept of binary operation. In addition, we examine some effects of a raw binary operation on the characteristics of each induced topology.
Keywords: Raw binary operation, left operand topology, right operand topology, output topology.

1. Introduction

Let's start with some basic definition [4–6]. Let $X \neq \emptyset$. We denote $X \times X$ by X^2, and also a graph of a binary relation F by $\Gamma(F)$. A (partial) binary operation (or simply binop) on X is a (partial) map from X^2 to X. A group-like structure consists of a nonempty set X, a (partial) binop f on X, and a finite number of specific axioms such as totality (closureness), associativity, identity, invertibility, and commutativity that f must satisfy (see [1–3, 7, 10] for more information). Some group-like structures such as semigroupoids, small categories and groupoids have partial binop that is weaker than binop.

We now give a weaker definition than partial binop. A *raw binop* on X, denoted by $F : X^2 \mathrel{+\!\!<} X$, is a binary relation F over sets X^2 and X. From now on, we will always use symbols such as $*$, $\circ$, $+$ or $\cdot$ for raw binops. A *raw binary structure* (or simply a *raw bistruct)*, denoted by $(X,*)$ (or simply X), is a set X equipped with a raw binop $*$ on itself. If $*$ is a raw binop, then we perefer to write $z \in x*y$ instead of $(x,y)*z$, and denote by $x*y$ the set of all z in X satisfying $z \in x*y$. Given raw binary structure $(X,*)$, we call $*$ by *trivial* if $\Gamma(*) = X^2 \times X$ or $\Gamma(*) = \emptyset$.

We can ask the natural question of whether the raw binop on a set X creates a base on X. In this article, we show that our answer to this question is positive. For this, we first introduce our three basic definitions: left operand, right operand and output set. The well-known fact in set-theoretical topology that topologies can be generated by collections

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of subsets of sets points to the importance of these three concepts we have introduced [8, 9, 11]. To put it more explicitly, we will be using left operand, right operand and output sets when generating three topologies from a raw binary operation.

2. Left Operand, Right Operand and Output Sets

Definition 1. Let $(X,*)$ be a raw bistruct and $(a,b) \in X^2$.

(1) The *left operand set* of (a,b) with respect to $*$, denoted by $L_{(X,*)}(a,b)$, $L_*(a,b)$, or simply $L(a,b)$, is defined by $\{x \in X | b \in x * a\}$.
(2) The *right operand set* of (a,b) with respect to $*$, denoted by $R_{(X,*)}(a,b)$, $R_*(a,b)$, or simply $R(a,b)$, is defined by $\{x \in X | b \in a * x\}$.
(3) The *output set* of (a,b) with respect to $*$, denoted by $O_{(X,*)}(a,b)$, $O_*(a,b)$, or simply $O(a,b)$, is defined by $\{x \in X | x \in a * b\}$.

Proposition 2.1. *Let $* : X^2 \multimap X$ be a raw binary operation, and $a,b,c \in X$.*

(1) For every $x,y,z \in X$, $z \in O(x,y) \Leftrightarrow x \in L(y,z) \Leftrightarrow y \in R(x,z)$.
(2) (a) If $L(b,c) \neq \emptyset$, then there exists $x \in X$ such that $R(x,c) \neq \emptyset$ and $O(x,b) \neq \emptyset$.
(b) If $R(a,c) \neq \emptyset$, then there exists $x \in X$ such that $L(x,c) \neq \emptyset$, and $O(a,x) \neq \emptyset$.
(c) If $O(a,b) \neq \emptyset$, then there exists $x \in X$ such that $L(b,x) \neq \emptyset$ and $R(a,x) \neq \emptyset$.
(3) (a) If $L(b,c) \neq X$, then there exists $x \in X$ such that $R(x,c) \neq X$ and $O(x,b) \neq X$.
(b) If $R(a,c) \neq X$, then there exists $x \in X$ such that $L(x,c) \neq X$, and $O(a,x) \neq X$.
(c) If $O(a,b) \neq X$, then there exists $x \in X$ such that $L(b,x) \neq X$ and $R(a,x) \neq X$.
(4) (a) If $L(b,c) = \emptyset$, then $b \notin R(x,c)$ and $c \notin O(x,b)$ for every $x \in X$.
(b) If $R(a,c) = \emptyset$, then $a \notin L(x,c)$ and $c \notin O(a,x)$ for every $x \in X$.
(c) If $O(a,b) = \emptyset$, then $a \notin L(b,x)$ and $b \notin R(a,x)$ for every $x \in X$.
(5) (a) If $L(b,c) = X$, then $b \in R(x,c)$ and $c \in O(x,b)$ for every $x \in X$.
(b) If $R(a,c) = X$, then $a \in L(x,c)$ and $c \in O(a,x)$ for every $x \in X$.
(c) If $O(a,b) = X$, then $a \in L(b,x)$ and $b \in R(a,x)$ for every $x \in X$.

Proof. For (2), (3), (4) and (5), we give only the first proofs, since the others can be proved with similar arguments.

(1) Given arbitrary $a,b,c \in X$ such that $c \in O(a,b)$. Then $c \in a * b$ which implies $a \in L(b,c)$ and $b \in R(a,c)$.
(2) (a) Since $L(b,c) \neq \emptyset$, we choose $x_0 \in X$ satisfying $x \in L(b,c)$. It follows by Proposition 2.1(1) that $R(x_0,c) \neq \emptyset$ and $O(x_0,b) \neq \emptyset$.
(3) (a) Let x be an element of X satisfying $x \notin L(b,c)$ from the hypothesis that $L(b,c) \neq X$. From Proposition 2.1(1), $R(x,c) \neq X$ and $O(x,b) \neq X$ for some $x \in X$.
(4) (a) By the hypothesis, $L(b,c) = \emptyset$ which implies that $x \notin L(b,c)$ for every $x \in X$. By Proposition 2.1(1), $b \notin R(x,c)$ and $c \notin O(x,b)$ for every $x \in X$, as desired.

(5) (a) It follows by the hypothesis that $x \in L(b,c)$ for every $x \in X$. From Proposition 2.1(1), we see immediately that $b \in R(x,c)$ and $c \in O(x,b)$ for every $x \in X$.

□

The *restriction of a raw binop* $*$ on X to $A \subseteq X$, denoted by $*|_A$, is a raw binop on A defined by

$$x *|_A y = (x * y) \cap A \text{ for every } x, y \in A.$$

Also, a left operand set $L_{*|_A}(a,b)$ is called the *restriction* of the left operand set $L_*(a,b)$ to A. Similarly, we can give this definition for the right operand sets and the output sets.

Proposition 2.2. *If $* : X^2 \mathrel{+\!\!\!<} X$ is a raw binop and $A \subseteq X$, then, the left operand (respectively, right operand, output) set of (a,b) with respect to $*|_A$ is the intersection of the subset A and the left operand (respectively, right operand, output) set of (a,b) with respect to $*$.*

Proof. It can be easily seen from the definition of the *restriction of a raw binop.* □

3. Left Operand, Right Operand and Output Topologies

We can consider a collection of subsets of a set X as a subbase for a topology on X. If we have a raw binary structure, then we can apply this fact to the collection of all left operand sets, the one of all right operand sets, or the one of all output sets. So we give the following definition.

Definition 2. Let $(X,*)$ be a raw bistruct.

(1) The *left operand topology* induced by $(X,*)$, denoted by $\mathscr{L}_{(X,*)}$, $\mathscr{L}_*$ or simply $\mathscr{L}$, is a topology for which $\{L(a,b) \,|\, a,b \in X\}$ is a subbase.
(2) The *right operand topology* induced by $(X,*)$, denoted by $\mathscr{R}_{(X,*)}$, $\mathscr{R}_*$ or simply $\mathscr{R}$, is a topology for which $\{R(a,b) \,|\, a,b \in X\}$ is a subbase.
(3) The *output topology* induced by $(X,*)$, denoted by $\mathscr{O}_{(X,*)}$, $\mathscr{O}_*$ or simply $\mathscr{O}$, is a topology for which $\{O(a,b) \,|\, a,b \in X\}$ is a subbase.

Example 3.1. *Let $X = \{a,b,c,d\}$. Given a raw binop $* : X^2 \mathrel{+\!\!\!<} X$ defined as in the table below.*

$*$	a	b	c	d
a	$\emptyset$	$\emptyset$	$\{b\}$	$\{a,d\}$
b	$\{b\}$	$\{c\}$	$\emptyset$	$\{b\}$
c	$\emptyset$	$\emptyset$	$\{b\}$	$\{b,c\}$
d	$\{b\}$	$\{a,c\}$	$\{b\}$	$\{b\}$

Notice that the operations $a*a$, $a*b$, $b*c$, $c*a$ and $c*b$ are undefined while the operations $a*d$, $c*d$, and $d*b$ have multiple results. All left operand sets, all right *operand sets and all output sets for the raw bistruct $(X,*)$ are given in the following tables, respectively:*

L	a	b	c	d
a	$\emptyset$	$\{b,d\}$	$\emptyset$	$\emptyset$
b	$\{d\}$	$\emptyset$	$\{b,d\}$	$\emptyset$
c	$\emptyset$	$\{a,c,d\}$	$\emptyset$	$\emptyset$
d	$\{a\}$	$\{b,c,d\}$	$\{c\}$	$\{a\}$

R	a	b	c	d
a	$\{d\}$	$\{c\}$	$\emptyset$	$\{d\}$
b	$\emptyset$	$\{a,d\}$	$\{b\}$	$\emptyset$
c	$\emptyset$	$\{c,d\}$	$\{d\}$	$\emptyset$
d	$\{b\}$	$\{a,c,d\}$	$\{b\}$	$\emptyset$

O	a	b	c	d
a	$\emptyset$	$\emptyset$	$\{b\}$	$\{a,d\}$
b	$\{b\}$	$\{c\}$	$\emptyset$	$\{b\}$
c	$\emptyset$	$\emptyset$	$\{b\}$	$\{b,c\}$
d	$\{b\}$	$\{a,c\}$	$\{b\}$	$\{b\}$

Observe that the last table above exhibiting all output sets is as the same as the operation table. Moreover, these two tables are the same for any raw bistruct. Considering these tables, we obtain the left operand topology, the right operand topology and the output topology induced by the raw bistruct $(X,*)$ as

$$\mathscr{L} = \{\emptyset, X, \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\},$$

$$\mathscr{R} = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\},$$

and

$$\mathscr{O} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\},$$

respectively.

Proposition 3.2. *Let $(X,*)$ be a raw binary structure.*

(1) The left operand (respectively, right operand, output) topology is indiscrete iff $L(a,b)$ (respectively, $R(a,b)$, $O(a,b)$) is either X or $\emptyset$ for every $a,b \in X$.
(2) The topologies $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{O}$ are all indiscrete iff the raw binop $*$ is trivial.
(3) The left operand (respectively, right operand, output) topology is discrete iff each singleton subset of X is a intersection of finitely many of left operand (respectively, right operand, output) sets.

Proof. (1) We give only the proof for the left operand topology. The other proofs follow similar arguments to those used in here.
($\Rightarrow$) This part of the proof follows immediately from the definition of left operand topology.
($\Leftarrow$) Assume that $\mathscr{L}$ is not indiscrete. Then we can find a non-empty $\mathscr{L}$-open set G_0, which is a proper subset of X. So G_0 is a union of finite intersections of non-empty left operand sets at least one of which does not equal X. This contradicts the hypothesis, therefore the assumption that $\mathscr{L}$ is not indiscrete is false.

(2) ($\Rightarrow$) Assume that the raw binop $*$ is not trivial, that is, $\Gamma(*) \neq \emptyset$ and $\Gamma(*) \neq X$. Then there exists two ordered pair in $X^2 \times X$, one of which, say $((a,b),c)$, is in $\Gamma(*)$ and the other of which, say $((d,e),f)$, is not in $\Gamma(*)$. It follows that $L(b,c) \neq \emptyset$ and $R(d,f) \neq X$. By the hypothesis and Proposition 3.2(1), we have $L(b,c) = X$ and $R(d,f) = \emptyset$. This implies that $d \in L(b,c)$ and $b \notin R(d,f)$. Then $c \in O(d,b) \not\ni f$ which means $O(d,b)$ is both nonempty and proper subset of X, which is impossible. Thus the assumption that $*$ is not trivial has led to a contradiction. Consequently, $*$ is trivial.
($\Leftarrow$) The necessary condition can be easily proved using Definition 2.

(3) We give only the proof for the left operand topology. The other proofs follow similar arguments to those used below.
($\Rightarrow$) This part of the proof follows immediately from the definition of left operand topology.
($\Leftarrow$) Let x be an arbitrary point in X. By the hypothesis, there exists a finite subset A of X^2 such that $\bigcap_{(a,b)\in A} L(a,b) = \{x\}$. Then, since all finite intersections of members of the subbase $\{L(a,b) \mid a,b \in X\}$ form a base $\mathscr{B}$ for the left operand topology $\mathscr{L}$ on X, $\mathscr{B}$ contains all singleton subsets of X. Hence $\mathscr{L}$ is discrete.

□

Theorem 3.3. *If $(X,*)$ is a raw bistruct and $A \subseteq X$, then $\left(A, \mathscr{L}_{*|_A}\right)$ (respectively, $\left(A, \mathscr{R}_{*|_A}\right)$, $\left(A, \mathscr{O}_{*|_A}\right)$) is a subspace of $(X, \mathscr{L}_*)$ (respectively, $(X, \mathscr{R}_*)$, $(X, \mathscr{O}_*)$).*

Proof. We give only the proof for the left operand topology. The other proofs follow similar arguments to those used in here.
Let $(X,*)$ be a raw bistruct and let G be an arbitrary $\mathscr{L}_{*|_A}$-open set. Then, by the definition of the left operand topology, there exists a collection $\mathscr{F}$ of finite subsets of A^2 such that $G = \bigcup_{F\in\mathscr{F}} \bigcap_{(a,b)\in F} L_{*|_A}(a,b)$. It follows by Proposition 2.2 that

$$\begin{aligned} G &= \bigcup_{F\in\mathscr{F}} \bigcap_{(a,b)\in F} L_{*|_A}(a,b) \\ &= \bigcup_{F\in\mathscr{F}} \bigcap_{(a,b)\in F} \left(L_*(a,b) \cap A\right) \\ &= \bigcup_{F\in\mathscr{F}} \left(\bigcap_{(a,b)\in F} L_*(a,b) \right) \cap A \\ &= \left(\bigcup_{F\in\mathscr{F}} \bigcap_{(a,b)\in F} L_*(a,b) \right) \cap A. \end{aligned}$$

Set $H = \bigcup_{F\in\mathscr{F}} \bigcap_{(a,b)\in F} L_*(a,b)$. Then, by the Definition 2(1), H is a $\mathscr{L}_*$-open set such that $G = H \cap A$. Since G is an arbitrarily chosen $\mathscr{L}_{*|_A}$-open set, the proof is complete. □

Theorem 3.4. *Let $(X,*)$ be a raw bistruct. The collection of all left operand (respectively, right operand, output) sets is a base for a topology on X iff the following are provided:*

(1) for every point $x \in X$, there exists a left operand (respectively, right operand, output) set containing x, and

(2) if x is a point in the intersection U of a pair of left operand (respectively, right operand, output) sets, then there exists a left operand (respectively, right operand, output) set containing x and contained by U.

Proof. We give only the proof for the left operand topology. The other proofs follow similar arguments to those used in here. Let $(X,*)$ be a raw bistruct.

($\Rightarrow$): (1) Let x be an arbitrary point in X. For some subset $A \subseteq X^2$, $X = \bigcup_{(a,b)\in A} L(a,b)$ since $\{L(a,b) \,|\, a,b \in X\}$ is a base for a topology on X. Then, there exists an ordered pair (a,b) in A such that $x \in L(a,b)$.

(2) Let U be an intersection of two left operand sets. Given $x \in U$. It follows from the fact that $\{L(a,b) \,|\, a,b \in X\}$ is a base for a topology on X that $U = \bigcup_{(a,b)\in A} L(a,b)$ for some subset $A \subseteq X^2$. Then, there exists an ordered pair (a,b) in A such that $x \in L(a,b)$. Say (a_0,b_0). Thus $x \in L(a_0,b_0) \subseteq U$.

($\Leftarrow$): X is the union of all left operand sets since there exists a left operand set containing an arbitrary point x in X. Let U be an intersection of two arbitrary left operand sets and $x \in U$. Then, by the hypothesis, there exists an ordered pair $(a_x,b_x) \in X^2$ such that $x \in L(a_x,b_x) \subseteq U$. Therefore, the union of such left operand sets $L(a_x,b_x)$ gives U. Hence the proof is complete.

□

Theorem 3.5. *Let $*$ and $\circ$ be two raw binops on a set X. $\mathscr{L}_\circ$ is finer than $\mathscr{L}_*$ iff, for every $x \in X$ and every finite subset A of X^2, there exists a finite subset B of X^2 such that $x \in \bigcap_{(a,b)\in A} L_*(a,b)$ implies $x \in \bigcap_{(a,b)\in B} L_\circ(a,b) \subseteq \bigcap_{(a,b)\in A} L_*(a,b)$. Similar conditions also hold for right operand and output topologies.*

Proof. We give only the proof for the left operand topology. The other proofs follow similar arguments to those used in here. Let $*,\circ : X^2 \multimap X$ be two raw binops.

($\Rightarrow$): Let $x \in X$ and A be a finite subset of X^2 such that $x \in L_*(a,b)$ for every $(a,b) \in A$. Set $G = \bigcap_{(a,b)\in A} L_*(a,b)$. Since the collection of all left operand sets generates $\mathscr{L}_*$, $G \in \mathscr{L}_*$. By the hypothesis, it also holds that $G \in \mathscr{L}_\circ$. Then, since the collection of finite intersections of left operand sets with respect to $*$ is a base for $\mathscr{L}_\circ$ and G is a $\mathscr{L}_\circ$-open set, there exists a finite subset $B \subseteq X^2$ such that $x \in \bigcap_{(a,b)\in B} L_\circ(a,b) \subseteq G$.

($\Leftarrow$): It's necessary to show that every $\mathscr{L}_*$-open set G is also a $\mathscr{L}_\circ$-open set. Let x be an arbitrary element of G. It follows immediately from the way the left operand topology $\mathscr{L}_*$ is generated that $x \in \bigcap_{(a,b)\in A} L_*(a,b) \subseteq G$ holds for some finite subset $A \subseteq X^2$. Then, by the hypothesis, there exists a finite subset $B \subseteq X^2$

such that $x \in \bigcap_{(a,b)\in B} L_{\circ}(a,b) \subseteq \bigcap_{(a,b)\in A} L_{*}(a,b) \subseteq G$. Since x is arbitrary, $G \in \mathscr{L}_{\circ}$ holds.

□

4. Further Work

We plan our next work to be a detailed examination of the separation axioms in each of the left operand, right operand, and output topologies generated by a raw binop. Immediately after, our study is to be a detailed study of the characteristics of each topological space created by a raw binop when it satisfies a certain one of the group-like axioms such as totality (closureness), associativity, identity, invertibility, and commutativity.

5. Conclusion

Some useful results obtained in this work can be summarized as follows: The left operand (respectively, right operand, output) topology generated by a restriction of a raw binop is a subspace of the left operand (respectively, right operand, output) topology generated by this raw binop. We have given the necessary and sufficient conditions for collections of the left operand (respectively, right operand, output) sets to be the base for a topology. We have presented the conditions for the two left operand (respectively, right operand, output) topologies generated by two raw binops defined on it to be comparable.

Competing interests

The author declares that they have no competing interests.

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