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FRACTIONAL INEQUALITIES OF MILNE-TYPE FOR TWICE DIFFERENTIABLE QUASI−CONVEX AND $P-$ FUNCTIONS

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Abstract. In this paper, a fundamental identity concerning twice differentiable functions is utilized as a key analytical tool. Utilizing this identity in the context of Riemann–Liouville fractional integrals, we establish novel Milne-type inequalities of fractional nature for functions whose second derivatives exhibit quasi-convexity and satisfy the conditions of $P-$ function class. The methodology further involves the application of classical Hölder and Young inequalities, leading to diverse and original results in the context of these generalized convexity concepts. The outcomes presented in this study not only enrich the theoretical framework of quasi-convex and $P-$functions but also contribute novel perspectives and techniques to the field of fractional calculus and integral inequalities.

Keywords: Milne-type inequalities, Quasi−convex, $P-$Functions, Fractional integrals.

1. INTRODUCTION

Numerical integration plays a central role in modern computational mathematics, particularly when the exact evaluation of integrals becomes impractical due to the complexity of the functions involved. In such cases, numerical methods offer reliable alternatives to approximate definite integrals. A major focus of current research lies in enhancing the accuracy of these methods and establishing more refined estimates for their error bounds. This necessitates a thorough understanding of the behavior of numerical integration errors, often achieved through the application of functional inequalities.

In this framework, special attention is given to function classes such as convex, bounded, and Lipschitzian functions, as they offer structural characteristics that directly influence error estimation. Of particular interest are functions whose first or second derivatives exhibit convexity, as these allow for the derivation of tighter and more informative upper bounds

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on numerical integration errors. The inherent geometric and analytical properties of such functions play a significant role in both theoretical investigations and practical applications.

This paper aims to provide a focused examination of various numerical integration techniques and their associated error bounds, with an emphasis on how function properties affect the performance and precision of these methods. By exploring classical and contemporary approaches, the study seeks to offer a deeper understanding of error behavior and to highlight pathways for further improvement in numerical integration practices.

1. The expression presented below corresponds to Simpsons quadrature formula, frequently referred to in the literature as Simpsons one-third rule due to the characteristic distribution of weights. This rule is widely used in numerical integration for approximating the definite integral of a function over a closed interval. It is derived by fitting a second-degree (quadratic) polynomial through three equally spaced points and integrating this interpolating polynomial instead of the original function. The standard form of Simpsons $\frac{1}{3}$ rule is given by:

$$\int_{\varkappa_1}^{\varkappa_2} \Omega(\hbar)d\hbar \approx \frac{\varkappa_2-\varkappa_1}{6}[\Omega(\varkappa_1)+4\Omega(\frac{\varkappa_1+\varkappa_2}{2})+\Omega(\varkappa_2)] \tag{1}$$

where Ω is assumed to be sufficiently smooth on $[\varkappa_1,\varkappa_2]$. This formula provides a higher degree of accuracy compared to the trapezoidal rule, especially when the integrand is well-approximated by a quadratic polynomial within the integration interval.

2. Also known as the NewtonCotes rule of degree three, Simpsons $\frac{3}{8}$ formula is presented as follows (see [10]):

$$\int_{\varkappa_1}^{\varkappa_2} \Omega(\hbar)d\hbar \approx \frac{\varkappa_2-\varkappa_1}{8}[\Omega(\varkappa_1)+3\Omega(\frac{2\varkappa_1+\varkappa_2}{3})+3\Omega(\frac{\varkappa_1+2\varkappa_2}{3})+\Omega(\varkappa_2)] \tag{2}$$

Equations (1) and (2) hold true under the assumption that the function Ω is four times continuously differentiable on the closed interval $[\varkappa_1,\varkappa_2]$, which guarantees the requisite regularity for the derivations involved.

Simpsons inequality can be stated in its classical form as follows.

Theorem 1.1. *Let us consider $\Omega:[\varkappa_1,\varkappa_2]\to\mathbb{R}$, a function with four continuous derivatives within the interval $(\varkappa_1,\varkappa_2)$, and $\left\|\Omega^{(4)}\right\|_\infty=\sup_{\hbar\in(\varkappa_1,\varkappa_2)}|\Omega^{(4)}(\hbar)|<\infty$, the subsequent inequality holds:*

$$\left|\frac{1}{6}\left[\Omega(\varkappa_1)+4\Omega(\frac{\varkappa_1+\varkappa_2}{2})+\Omega(\varkappa_2)\right]-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar\right|\le\frac{1}{2880}\left\|\Omega^{(4)}\right\|_\infty(\varkappa_2-\varkappa_1)^4.$$

Simpson-type inequalities were first introduced by Sarkaya and collaborators [11], employing the concept of convexity as the foundational framework. In later developments, these inequalities were adapted to the setting of RiemannLiouville fractional integrals, leading to the classification of three main variants, each based on distinct formulations involving fractional operators.These variations have been thoroughly developed and analyzed in [12]-[14], thereby extending Simpsons inequality within the framework of fractional analysis and demonstrating its applicability across diverse types of fractional integrals. Notable contributions to this line of research can be found in [16]-[24], where the classical Simpson inequality is extended within the scope of fractional calculus and applied to various integral types.

Further refinements have been proposed in works such as [15][19], which focus specifically on Simpson-type inequalities for functions with continuous second-order derivatives. These studies provide in-depth analysis of how such smoothness conditions influence the structure and sharpness of the resulting inequalities. As a result, the applicability of Simpson-type inequalities has significantly expanded, enabling more specialized and accurate estimates tailored to diverse function classes.

The Newton inequality, another classical result in mathematical analysis, remains a cornerstone in the study of integral approximations and can be described in the following standard form:

Theorem 1.2. *([10]) Let $\Omega : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a function whose fourth derivative exists and is continuous on the open interval $(\varkappa_1, \varkappa_2)$, and assume that $\left\|\Omega^{(4)}\right\|_\infty = \sup_{\hbar \in (\varkappa_1, \varkappa_2)} |\Omega^{(4)}(\hbar)| < \infty$. Under these conditions, the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{8}\left[\Omega(\varkappa_1) + 3\Omega(\frac{2\varkappa_1 + \varkappa_2}{3}) + 3\Omega(\frac{\varkappa_1 + 2\varkappa_2}{3}) + \Omega(\varkappa_2)\right] - \frac{1}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \Omega(\hbar)d\hbar\right| \\ \leq\ & \frac{1}{6480}\left\|\Omega^{(4)}\right\|_\infty (\varkappa_2 - \varkappa_1)^4. \end{aligned}$$

References [20][22] present the formulation of Newton-type inequalities within the framework of convex functions applied to local fractional integrals. These contributions extend the scope of Newton's inequality, offering novel insights in the area of fractional analysis. In [38], foundational results were established through the first demonstrations of Newton-type inequalities for RiemannLiouville fractional integrals, marking a significant milestone and serving as a key reference for subsequent investigations in fractional calculus. Subsequent works, such as [23, 24], have focused on the development of Newton-type inequalities within the framework of RiemannLiouville fractional integrals, offering detailed analyses of their validity under various mathematical settings. Together, these contributions have significantly enriched the theory of fractional integration and helped bridge important gaps in the existing body of research.

Milnes classical inequality represents a key result in mathematical analysis and appears in various applied contexts. Its standard form is given below.

Theorem 1.3. *([25]) Assume that $\Omega : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a function whose fourth derivative exists and is continuous on the interval $(\varkappa_1, \varkappa_2)$, and suppose that $\left\|\Omega^{(4)}\right\|_\infty = \sup_{\hbar \in (\varkappa_1, \varkappa_2)} |\Omega^{(4)}(\hbar)| < \infty$. Under these assumptions, the inequality below holds true:*

$$\left|\frac{1}{3}\left[2\Omega(\varkappa_1) - \Omega(\frac{\varkappa_1+\varkappa_2}{2}) + 2\Omega(\varkappa_2)\right] - \frac{1}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \Omega(\hbar)d\hbar\right| \leq \frac{7(\varkappa_2-\varkappa_1)^4}{23040}\left\|\Omega^{(4)}\right\|_\infty$$

The initial formulation of Milne-type inequalities grounded in convexity theory was introduced by Djenaoui and Meftah in [26], marking a notable step forward in the analytical treatment of these inequalities from a convex perspective. Their findings offered a broader theoretical interpretation of Milne-type results. Building on this foundation, Budak and co-authors in [27] extended the applicability of such inequalities by embedding them within the context of RiemannLiouville fractional integrals. This advancement contributed significantly to the structural development of fractional calculus, enhancing the utility of Milne-type inequalities in diverse mathematical settings.

More recent investigations, including those in [28, 29], have proposed new variants of fractional Milne-type inequalities, with particular attention to separable convex functions. These efforts have further expanded the reach of such inequalities by analyzing their behavior over a variety of function classessuch as bounded, Lipschitz continuous, and functions of bounded variation. For detailed theoretical background and a broader overview of applications and formulations, the reader is referred to studies such as [30, 31, 32, 40, 41, 42, 43], which provide comprehensive insights into the structure and scope of Milne-type inequalities.

In this paper, we aim to develop a set of novel Milne-type inequalities within the framework of fractional calculus, specifically by employing the RiemannLiouville fractional integral operators. Our focus is directed toward functions whose second-order derivatives fulfill convexity-related conditions, such as quasi-convexity and membership in the class of P-functions. To set the stage, we provide a comprehensive overview of the relevant mathematical tools and function classes that play a central role in the derivation process. The established results not only generalize classical Milne-type inequalities but also offer deeper insights into the interplay between convexity properties and fractional integral operators. It is anticipated that these findings will contribute meaningfully to the ongoing development of integral inequalities in the fractional setting.

Definition 1. ([8]) Let $\Psi \in L_1[\hbar_1, \hbar_2]$, with $\alpha > 0$ and $\hbar_1 \geq 0$. Then, the left-sided and right-sided Riemann–Liouville fractional integrals of order α are defined as follows:

$$J_{\hbar_1^+}^{\alpha}\Psi(u_1) = \frac{1}{\Gamma(\alpha)}\int_{\hbar_1}^{u_1}(u_1 - \flat)^{\alpha-1}\Psi(\flat)d\flat, \quad u_1 > \hbar_1$$

and

$$J^{\alpha}_{\hbar_2^-}\Psi(u_1)=\frac{1}{\Gamma(\alpha)}\int_{u_1}^{\hbar_2}(\flat-u_1)^{\alpha-1}\Psi(\flat)d\flat,\qquad u_1<\hbar_2$$

where the Gamma function $\Gamma(\alpha)$ is given by

$$\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}\,du,$$

and it is understood that the fractional integrals of order zero satisfy

$$J^{0}_{\hbar_1^+}\Psi(u_1)=J^{0}_{\hbar_2^-}\Psi(u_1)=\Psi(u_1).$$

When the parameter α is assigned the value 1, the defined fractional integral operator naturally collapses to the classical (first-order) integral, thereby maintaining alignment with the conventional integral formulation. This limiting case highlights the consistency of the fractional framework with classical analysis. Over the years, a substantial body of research has emerged examining the structural characteristics, theoretical foundations, and diverse applications of such fractional operators; for comprehensive treatments, see, for example, [2][8].

Next, we proceed by presenting the definitions of convexity, quasi−convexity, and $P-$functions.

Definition 2. ([9]) Let $I\subset\mathbb{R}$ be an interval. A function $\Omega:I\to\mathbb{R}$ is said to be *convex* if, for any $\varkappa_1,\varkappa_2\in I$ and for every $\hbar\in[0,1]$, the following condition is satisfied:

$$\Omega(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\leq\hbar\Omega(\varkappa_1)+(1-\hbar)\Omega(\varkappa_2).$$

Definition 3. ([33]) A function $\Omega:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is said to be quasi-convex on $[\varkappa_1,\varkappa_2]$ if

$$\Omega(\hbar\rho+(1-\hbar)\rho)\leq\max\{\Omega(\rho),\Omega(\rho)\}\,,\ \text{for all }\rho,\rho\in[\varkappa_1,\varkappa_2]\,.$$

The relationship between the concepts of convexity and quasi-convexity can be found in the papers [34]-[38].

Definition 4. ([39]) Let $\Omega:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a function. We say that Ω is a member of the class of P-functions if Ω is nonnegative on $[\varkappa_1,\varkappa_2]$ and satisfies the inequality

$$\Omega(\hbar\rho+(1-\hbar)\rho)\leq\Omega(\rho)+\Omega(\rho)$$

for all $\rho,\rho\in[\varkappa_1,\varkappa_2]$ and $\hbar\in[0,1]$.

Lemma 1.4. *([1]) If $\Omega:[\varkappa_1,\varkappa_2]\longrightarrow R$ is absolutely continuous over $(\varkappa_1,\varkappa_2)$ and $\Omega''\in L_1([\varkappa_1,\varkappa_2])$, then the following holds:*

$$\begin{aligned}&\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\\=&\ \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\sum_{k=1}^{4}I_k,\end{aligned}$$

where

$$I_1=\int_0^{\frac{1}{2}}(\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar)\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)d\hbar,$$
$$I_2=\int_0^{\frac{1}{2}}(\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar)\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)d\hbar,$$
$$I_3=\int_{\frac{1}{2}}^1(\hbar^{\flat+1}-\hbar)\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)d\hbar,$$
$$I_4=\int_{\frac{1}{2}}^1(\hbar^{\flat+1}-\hbar)\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)d\hbar.$$

2. Main Result

2.1. Fractional Inequalities of Milne-Type for Twice Differentiable Quasi−convex.

Theorem 2.1. *Assuming that the conditions specified in Lemma 1.4 are fulfilled, and additionally that the function $|\Omega''|$ is quasi-convex on the interval $[\varkappa_1,\varkappa_2]$, the following inequality is established:*

$$\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2(\flat^2+15\flat+2)}{24(\flat+2)(\flat+1)}\left(max\left\{\left|\Omega''(\varkappa_1)\right|,\left|\Omega''(\varkappa_2)\right|\right\}\right)$$

Proof. Utilizing Lemma 1.4 under the modulus operation framework leads to the inequality given below:

$$\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\right.$$
$$+\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar$$
$$+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar$$
$$\left.+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\right].$$

Leveraging the Quasi-convex property of $\left|\Omega''\right|$, we derive

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)\left[max\left\{\left|\Omega''(\varkappa_2)\right|,\left|\Omega''(\varkappa_1)\right|\right\}\right]d\hbar\right. \\
&+\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)\left[max\left\{\left|\Omega''(\varkappa_1)\right|,\left|\Omega''(\varkappa_2)\right|\right\}\right]d\hbar \\
&+\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)\left[max\left\{\left|\Omega''(\varkappa_2)\right|,\left|\Omega''(\varkappa_1)\right|\right\}\right]d\hbar \\
&\left.+\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)\left[max\left\{\left|\Omega''(\varkappa_1)\right|,\left|\Omega''(\varkappa_2)\right|\right\}\right]d\hbar\right]. \\
&=\frac{(\varkappa_2-\varkappa_1)^2(\flat^2+15\flat+2)}{24(\flat+2)(\flat+1)}\left(max\left\{\left|\Omega''(\varkappa_1)\right|,\left|\Omega''(\varkappa_2)\right|\right\}\right).
\end{aligned}$$

Corollary 2.2. *By taking $\flat=1$ in Theorem 2.1, the subsequent inequality is derived.*

$$\begin{aligned}
&\left|\frac{1}{(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
&\leq\frac{(\varkappa_2-\varkappa_1)^2}{8}\left[max\left\{\left|\Omega''(\varkappa_1)\right|,\left|\Omega''(\varkappa_2)\right|\right\}\right]
\end{aligned}$$

□

Theorem 2.3. *Suppose that the hypotheses of Lemma 1.4 hold. Furthermore, if $|\Omega''|^q$, for some $q>1$, is m-convex on the interval $[\varkappa_1,\varkappa_2]$, then the following result holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}+\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^\flat\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[\frac{max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Here, the parameters p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$*, and* $\mathscr{B}$ *denotes the incomplete Beta function, which is given by:*

$$\mathscr{B}(n,m,v)=\int_0^v \hbar^{n-1}(1-\hbar)^{m-1}\,d\hbar.$$

Proof. Utilizing Lemma 1.4 under the modulus operation framework leads to the inequality given below:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right|\\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\right.\\
&+\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\\
&+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\\
&\left.+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\right].
\end{aligned}$$

Utilizing Hlder's inequality on the preceding inequality leads to the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right|\\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the Quasi-convex function of $\left|\Omega''\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)^{\frac{1}{q}}\right] \\
&=\frac{(\varkappa_2-\varkappa_1)^2}{(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}+\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^\flat\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[\frac{max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

□

Corollary 2.4. *By taking $\flat=1$ in Theorem 2.3, the subsequent inequality is derived.*

$$\begin{aligned}
&\left|\frac{1}{(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\hbar-\hbar^2\right)^p d\hbar\right)^{\frac{1}{p}}+\left(\mathscr{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[\frac{max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Here, the parameters p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$, and $\mathscr{B}$ denotes the incomplete Beta function, which is given by:

$$\mathscr{B}(n,m,v)=\int_0^v \hbar^{n-1}(1-\hbar)^{m-1}d\hbar.$$

Theorem 2.5. *Assume the conditions of Lemma 1.4 are satisfied. If $\left|\Omega''\right|^q$ is quasi−convex function over the interval $[\varkappa_1,\varkappa_2]$, then:*

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{(\flat+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{p}\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^\flat\right)\right)\right. \\
&\left.+\frac{max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}}{q}\right]
\end{aligned}$$

where $q>1$.
Here, the parameters p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$, and $\mathscr{B}$ denotes the incomplete Beta function, which is given by:

$$\mathscr{B}(n,m,v)=\int_0^v \hbar^{n-1}(1-\hbar)^{m-1}\,d\hbar.$$

Proof. Utilizing Lemma 1.4 under the modulus operation framework leads to the inequality given below:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\right. \\
&+\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar \\
&+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar \\
&\left.+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\right].
\end{aligned}$$

Utilizing Young's inequality on the preceding inequality leads to the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)\right.\\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)\\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)\\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)\right].
\end{aligned}$$

Utilizing the quasi−convex function of $\left|\Omega''\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)\right.\\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)\\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1 max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)\\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1 max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}d\hbar\right)\right]\\
&=\frac{(\varkappa_2-\varkappa_1)^2}{(\flat+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{p}\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^{\flat}\right)\right)\right.\\
&\left.+\frac{max\left\{\left|\Omega''(\varkappa_1)\right|^q,\left|\Omega''(\varkappa_2)\right|^q\right\}}{q}\right].
\end{aligned}$$

□

Remark 2.6. *By taking $\flat = 1$ in Theorem 2.5, the following inequality is derived:*

$$\begin{aligned}
&\left|\frac{1}{(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar - \frac{1}{3}\left[2\Omega(\varkappa_1) - \Omega(\frac{\varkappa_1+\varkappa_2}{2}) + 2\Omega(\varkappa_2)\right]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2} \times \left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\hbar - \hbar^2\right)^p d\hbar\right) + \frac{1}{p}\left(\mathscr{B}\left(p+1, p+1, \frac{1}{2}\right)\right)\right. \\
&+ \left.\frac{max\left\{\left|\Omega''(\varkappa_1)\right|^q, \left|\Omega''(\varkappa_2)\right|^q\right\}}{q}\right].
\end{aligned}$$

2.2. Fractional Inequalities of Milne-Type for Twice Differentiable P−Functions.

Theorem 2.7. *Let the assumptions of Lemma 1.4 hold. In addition, suppose that $|\Omega''|$ belongs to the class of P-functions on the interval $[\varkappa_1, \varkappa_2]$. Then, the following inequality is satisfied:*

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2) + \Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)] - \frac{1}{3}[2\Omega(\varkappa_1) - \Omega(\frac{\varkappa_1+\varkappa_2}{2}) + 2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2(\flat^2+15\flat+2)}{24(\flat+2)(\flat+1)}\left(\left|\Omega''(\varkappa_1)\right| + \left|\Omega''(\varkappa_2)\right|\right)
\end{aligned}$$

Proof. Utilizing Lemma 1.4 under the modulus operation framework leads to the inequality given below:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2) + \Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)] - \frac{1}{3}[2\Omega(\varkappa_1) - \Omega(\frac{\varkappa_1+\varkappa_2}{2}) + 2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1} - \frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_2 + (1-\hbar)\varkappa_1)\right|d\hbar\right. \\
&+ \int_0^{\frac{1}{2}}\left|\hbar^{\flat+1} - \frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_1 + (1-\hbar)\varkappa_2)\right|d\hbar \\
&+ \int_{\frac{1}{2}}^1\left|\hbar^{\flat+1} - \hbar\right|\left|\Omega''(\hbar\varkappa_2 + (1-\hbar)\varkappa_1)\right|d\hbar \\
&+ \left.\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1} - \hbar\right|\left|\Omega''(\hbar\varkappa_1 + (1-\hbar)\varkappa_2)\right|d\hbar\right].
\end{aligned}$$

Leveraging the $P-$functions property of $\left|\Omega''\right|$, we derive

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right)d\hbar\right. \\
&+\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right)d\hbar \\
&+\int_{\frac{1}{2}}^{1}\left(\hbar-\hbar^{\flat+1}\right)\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right)d\hbar \\
&\left.+\int_{\frac{1}{2}}^{1}\left(\hbar-\hbar^{\flat+1}\right)\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right)d\hbar\right]. \\
&=\frac{(\varkappa_2-\varkappa_1)^2(\flat^2+15\flat+2)}{24(\flat+2)(\flat+1)}\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right).
\end{aligned}$$

Corollary 2.8. *By taking $\flat=1$ in Theorem 2.7, the subsequent inequality is derived.*

$$\begin{aligned}
&\left|\frac{1}{(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
&\leq\frac{(\varkappa_2-\varkappa_1)^2}{8}\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right)
\end{aligned}$$

□

Theorem 2.9. *Suppose that the hypotheses of Lemma 1.4 are satisfied. Moreover, let $q>1$ and assume that the function $|\Omega''|^q$ belongs to the class of P-functions on the interval $[\varkappa_1,\varkappa_2]$. Under these conditions, we have:*

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}+\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^{\flat}\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[\frac{\left(\left|\Omega''(\varkappa_1)\right|+\left|\Omega''(\varkappa_2)\right|\right)}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Here, the parameters p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$*, and* $\mathscr{B}$ *denotes the incomplete Beta function, which is given by:*

$$\mathscr{B}(n,m,v)=\int_0^v \hbar^{n-1}(1-\hbar)^{m-1}\,d\hbar.$$

Proof. Utilizing Lemma 1.4 under the modulus operation framework leads to the inequality given below:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\mathfrak{I}^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\mathfrak{I}^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right|\\
\leq\;&\frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\right.\\
&+\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\\
&+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\\
&\left.+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\right].
\end{aligned}$$

Utilizing Hlder's inequality on the preceding inequality leads to the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\mathfrak{I}^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\mathfrak{I}^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right|\\
\leq\;&\frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the $P-$functions of $\left|\Omega''\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q\right)d\hbar\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q\right)d\hbar\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q\right)d\hbar\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q\right)d\hbar\right)^{\frac{1}{q}}\right] \\
&=\frac{(\varkappa_2-\varkappa_1)^2}{(\flat+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)^{\frac{1}{p}}+\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^\flat\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[\frac{\left(\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q\right)}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 2.10. *By taking $\flat=1$ in Theorem 2.9, the subsequent inequality is derived.*

$$\begin{aligned}
&\left|\frac{1}{(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\hbar-\hbar^2\right)^p d\hbar\right)^{\frac{1}{p}}+\left(\mathscr{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[\frac{\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Here, the parameters p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$, and $\mathscr{B}$ denotes the incomplete Beta function, which is given by:

$$\mathscr{B}(n,m,v)=\int_0^v\hbar^{n-1}(1-\hbar)^{m-1}d\hbar.$$

Theorem 2.11. *Assume the conditions of Lemma 1.4 are satisfied, if* $\left|\Omega''\right|^q$ *is* $P-$*function over the interval* $[\varkappa_1,\varkappa_2]$, *then :*

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{(\flat+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{p}\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^\flat\right)\right)\right. \\
&\left.+\frac{\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q}{q}\right]
\end{aligned}$$

where $q>1$. *Here, the parameters* p *and* q *satisfy the relation* $\frac{1}{p}+\frac{1}{q}=1$, *and* $\mathscr{B}$ *denotes the incomplete Beta function, which is given by:*

$$\mathscr{B}(n,m,v)=\int_0^v \hbar^{n-1}(1-\hbar)^{m-1}\,d\hbar.$$

Proof. Utilizing Lemma 1.4 under the modulus operation framework leads to the inequality given below:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^\flat}[\Im^\flat_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^\flat_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar\right. \\
&+\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar \\
&+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|d\hbar \\
&\left.+\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|d\hbar\right].
\end{aligned}$$

Utilizing Young's inequality on the preceding inequality leads to the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\; & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\hbar^{\flat+1}-\frac{\flat+4}{3}\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_2+(1-\hbar)\varkappa_1)\right|^q d\hbar\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\hbar^{\flat+1}-\hbar\right|^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\hbar\varkappa_1+(1-\hbar)\varkappa_2)\right|^q d\hbar\right)\right].
\end{aligned}$$

Utilizing the definition of $\left|\Omega''\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\flat+1)}{2(\varkappa_2-\varkappa_1)^{\flat}}[\Im^{\flat}_{\varkappa_1^+}\Omega(\varkappa_2)+\Im^{\flat}_{\varkappa_2^-}\Omega(\varkappa_1)]-\frac{1}{3}[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)]\right| \\
\leq\; & \frac{(\varkappa_2-\varkappa_1)^2}{2(\flat+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q d\hbar\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q d\hbar\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q d\hbar\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q d\hbar\right)\right] \\
&=\frac{(\varkappa_2-\varkappa_1)^2}{(\flat+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\flat+4}{3}\hbar-\hbar^{\flat+1}\right)^p d\hbar\right)+\frac{1}{p}\left(\frac{1}{\flat}\mathscr{B}\left(p+1,\frac{p+1}{\flat},1-\left(\frac{1}{2}\right)^{\flat}\right)\right)\right. \\
&\left.+\frac{\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q}{q}\right].
\end{aligned}$$

□

Corollary 2.12. *By taking $\flat = 1$ in Theorem 2.11, the following inequality is derived:*

$$\left|\frac{1}{(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\Omega(\hbar)d\hbar-\frac{1}{3}\left[2\Omega(\varkappa_1)-\Omega(\frac{\varkappa_1+\varkappa_2}{2})+2\Omega(\varkappa_2)\right]\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{2}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\hbar-\hbar^2\right)^p d\hbar\right)+\frac{1}{p}\left(\mathscr{B}\left(p+1,p+1,\frac{1}{2}\right)\right)\right.$$
$$\left.+\frac{\left|\Omega''(\varkappa_1)\right|^q+\left|\Omega''(\varkappa_2)\right|^q}{q}\right].$$

Here, the parameters p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$, and $\mathscr{B}$ denotes the incomplete Beta function, which is given by:

$$\mathscr{B}(n,m,v)=\int_0^v \hbar^{n-1}(1-\hbar)^{m-1}\,d\hbar.$$

Competing interests

The authors declare that they have no competing interests.

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Author's contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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