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POSITIVITY OF SUMS AND INTEGRALS FOR $n-$CONVEX FUNCTIONS VIA EXTENSION OF MONTGOMERY IDENTITY USING NEW GREEN FUNCTIONS

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Abstract. New general linear (integral and discrete) identities and inequalities are given for convex functions of order n via extension of Montgomery identity using new Green functions. We also state positivity conditions for these inequalities. We also study n-convexity at a point for our proposed inequalities. Bounds for reminders for proposed results are also given by using Grüss- and Ostrowski-types inequalities. We would also state mean value results of Cauchy type and Lagrange type.

Keywords: Convex functions of order n, Montgomery identity, Čebyšev functional, Green functions, n-convexity at a point.

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1. Introduction

In [8], authors proved various results related to general linear inequalities via extended Montgomery identity using Green's function (see also [10]). Recently, in [3] authors have introduced new Green type functions. Our main objective of present article is to further extend results of [8] using new definitions stated in [3].

To recall definitions of generalized convex function and related concepts and results we refer to interested readers following references [6], [9] and [18].

In further text we would use notation $AC(I)$ for class of absolutely continuous functions defined on a real interval I and by $(\xi - s)_+^k$, $k \in \mathbb{N}$, we would mean following

$$(\xi - s)_+^k = \begin{cases} (\xi - s)^k, & \text{if } \xi \geq s \\ 0, & \text{if } \xi < s. \end{cases}$$

We also denote $\{1, 2, \ldots, m\}$ by I_m.

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Popoviciu proved following result [19, 20] (see [17, 18, 22]).

Proposition 1.1. *Let $n \geq 2$. Inequality*

$$\sum_{i \in I_m} \omega_i f(\eta_i) \geq 0 \tag{1.1}$$

is valid for all convex functions of order n, $f : [a,b] \to \mathbb{R}$ iff $m-$tuples $\eta \in [a,b]^m$, $\omega \in \mathbb{R}^m$ satisfy

$$\sum_{i \in I_m} \omega_i \eta_i^k = 0, \quad \forall\, k \in I_{n-1} \tag{1.2}$$

$$\sum_{i \in I_m} \omega_i (\eta_i - \tau)_+^{n-1} \geq 0, \quad \forall\, \tau \in [a,b]. \tag{1.3}$$

Remark 1.2. Case $n = 2$ was of particular interest and in [14] (see also [18, p.262]) it is proved that if $n = 2$ conditions (1.2) and (1.3) can be replaced with

$$\sum_{i \in I_m} \omega_i = 0 \quad \text{and} \quad \sum_{i \in I_m} \omega_i |\eta_i - \eta_k| \geq 0 \text{ for } k \in I_m. \tag{1.4}$$

The integral analogue of previous result is stated here as (see [17, 18, 21]).

Proposition 1.3. *Let $2 \leq n \in \mathbb{N}$, $\omega : [\alpha, \beta] \to \mathbb{R}$ and $\lambda : [\alpha, \beta] \to [a,b]$. Then, inequality*

$$\int_\alpha^\beta \omega(\eta) f(\lambda(\eta))\, d\eta \geq 0 \tag{1.5}$$

is valid for all convex functions of order n, $f : [a,b] \to \mathbb{R}$ iff

$$\begin{aligned} \int_\alpha^\beta \omega(\eta) \lambda(\eta)^k\, d\eta = 0, \quad \forall\, k \in I_{n-1} \\ \int_\alpha^\beta \omega(\eta) \left(\lambda(\eta) - t\right)_+^{n-1} d\eta \geq 0, \quad \forall\, t \in [a,b]. \end{aligned} \tag{1.6}$$

In article [1] following extension of Montgomery identity is given.

Proposition 1.4. *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)} \in AC(I)$ for $2 \leq n \in \mathbb{N}$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Then following identity is valid*

$$\begin{aligned} f(\eta) = \frac{1}{b-a} \int_a^b f(s)\, ds + \sum_{k \in \{0,\ldots,n-2\}} \frac{f^{(k+1)}(a)}{k!\,(k+2)} \frac{(\eta - a)^{k+2}}{b-a} \\ - \sum_{k \in \{0,\ldots,n-2\}} \frac{f^{(k+1)}(b)}{k!\,(k+2)} \frac{(\eta - b)^{k+2}}{b-a} + \frac{1}{(n-1)!} \int_a^b T_n(\eta, \tau) f^{(n)}(\tau)\, d\tau \end{aligned} \tag{1.7}$$

where

$$T_n(\eta, \tau) = \begin{cases} -\frac{(\eta-\tau)^n}{n(b-a)} + \frac{\eta-a}{b-a} (\eta - \tau)^{n-1}, & a \leq \tau \leq \eta, \\ -\frac{(\eta-\tau)^n}{n(b-a)} + \frac{\eta-b}{b-a} (\eta - \tau)^{n-1}, & \eta < \tau \leq b. \end{cases} \tag{1.8}$$

Aim of the article is to present general linear identities and general linear inequalities for $n-$convex functions via extension of Montgomery identity using new Green functions. We also aim to study n-convexity at a point for our proposed results. We also obtain bounds for reminders in general linear inequalities by using Čebyšev functional, Ostrowski- and Gr̈uss- type inequalities. We would also state mean value theorems.

2. General linear identities and inequalities via extension of Montgomery identity and New Green Functions

In present section we obtain some discrete and integral identities and corresponding linear inequalities using new Green functions and applying Montgomery identity.

From [16] we recall some identities stated as under:

$$f(\eta) = f(a) + (\eta - a)f'(b) + \int_a^b G_1(\eta,\tau)f''(\tau)d\tau, \tag{2.1}$$

where $G_1(s,t)$ is Green's function is defined as

$$G_1(s,\tau) = \begin{cases} a-\tau, & a \le \tau \le s, \\ a-s, & s \le \tau \le b. \end{cases} \tag{2.2}$$

From [3] we recall some new types of Green functions $G_l : [a,b]\times[a,b] \to \mathbb{R}$, $(l \in \{2,3,4\},)$ defined as under:

$$G_2(s,\tau) = \begin{cases} s-b, & a \le \tau \le s, \\ \tau-b, & s \le \tau \le b. \end{cases} \tag{2.3}$$

$$G_3(s,\tau) = \begin{cases} s-a, & a \le \tau \le s, \\ \tau-a, & s \le \tau \le b. \end{cases} \tag{2.4}$$

$$G_4(s,\tau) = \begin{cases} b-\tau, & a \le \tau \le s, \\ b-s, & s \le \tau \le b. \end{cases} \tag{2.5}$$

In [3], it is also shown that all four Green functions are symmetric and continuous.

Lemma 2.1. *[3] Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function and G_l, $(l \in I_4)$ be new Green functions defined above. Then along with* (2.1) *following identities are valid:*

$$f(\eta) = f(b) + (b-\eta)f'(a) + \int_a^b G_2(\eta,\tau)f''(\tau)d\tau, \tag{2.6}$$

$$f(\eta) = f(b) - (b-a)f'(b) + (\eta-a)f'(a) + \int_a^b G_3(\eta,\tau)f''(\tau)d\tau, \tag{2.7}$$

$$f(\eta) = f(a) + (b-a)f'(a) - (b-\eta)f'(b) + \int_a^b G_4(\eta,\tau)f''(\tau)d\tau. \tag{2.8}$$

Proof. Consider integral

$$\int_a^b G_l(\eta,\tau)f''(\tau)d\tau = \int_a^x G_l(\eta,\tau)f''(\tau)d\tau + \int_x^b G_l(\eta,\tau)f''(\tau)d\tau.$$

Fix $l \in I_4$, and integration by parts, we would obtained identities (2.1), (2.6),(2.7) and (2.8) for $l = 1,2,3$ and 4 respectively. □

Now we can state main results of present section.

Theorem 2.2. *Fix $l \in I_4$. Let $\eta = (\eta_1,\dots,\eta_m) \in [a,b]^m$, $\omega = (\omega_1,\dots,\omega_m) \in \mathbb{R}^m$ satisfy conditions*

$$\sum_{i\in I_m} \omega_i = 0, \qquad \sum_{i\in I_m} \omega_i \eta_i = 0. \tag{2.9}$$

Also let $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)} \in AC(I)$ for $n \in \mathbb{N}$, $n \geq 3$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$, then for all $s \in [a,b]$ we have following identity

$$\sum_{i\in I_m} \omega_i f(\eta_i) = \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)ds$$
$$+\sum_{k=2}^{n-1} \frac{k}{(k-1)!}\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$+\frac{1}{(n-3)!}\int_a^b f^{(n)}(\tau)\left(\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)\tilde{T}_{n-2}(s,\tau)ds\right)d\tau \tag{2.10}$$

where

$$\tilde{T}_{n-2}(s,\tau) = \begin{cases} \frac{1}{b-a}\left[\frac{(s-\tau)^{n-2}}{(n-2)} + (s-a)(s-\tau)^{n-3}\right], & a \leq \tau \leq s \leq b, \\ \frac{1}{b-a}\left[\frac{(s-\tau)^{n-2}}{(n-2)} + (s-b)(s-\tau)^{n-3}\right], & a \leq s < \tau \leq b. \end{cases} \tag{2.11}$$

and G_l are as defined in (2.2) – (2.5). *Moreover, we also obtain following identity*

$$\sum_{i\in I_m} \omega_i f(\eta_i) = \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)ds$$
$$+\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!}\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$+\frac{1}{(n-3)!}\int_a^b f^{(n)}(\tau)\left(\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)T_{n-2}(s,\tau)ds\right)d\tau \tag{2.12}$$

where T_n is as defined in Proposition 1.4.

Proof. First consider four identities $(2.1),(2.6),(2.7)$ and (2.8), and putting $\eta=\eta_i$ in all these identities, multiplying each with ω_i, and then summing over each identity for $i\in I_m$ and using conditions that $\sum_{i\in I_m}\omega_i=0$, $\sum_{i\in I_m}\omega_i\eta_i=0$ we get by fixing $l\in I_4$

$$\sum_{i\in I_m}\omega_i f(\eta_i)=\int_a^b\left(\sum_{i\in I_m}\omega_i G_l(\eta_i,\tau)\right)f''(\tau)d\tau. \tag{2.13}$$

Differentiating (1.7) twice with respect to $\eta=s$, and then replacing k by $k-1$, we get

$$f''(s)=\frac{f'(a)-f'(b)}{b-a}+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}\\ +\frac{1}{(n-3)!}\int_a^b\tilde{T}_{n-2}(s,\tau)f^{(n)}(\tau)d\tau. \tag{2.14}$$

Now using (2.14) in (2.13) we get

$$\sum_{i\in I_m}\omega_i f(\eta_i)=\frac{f'(a)-f'(b)}{b-a}\int_a^b\sum_{i\in I_m}\omega_i G_l(\eta_i,s)ds\\ +\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i\in I_m}\omega_i G_l(\eta_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\\ +\frac{1}{(n-3)!}\int_a^b\sum_{i\in I_m}\omega_i G_l(\eta_i,s)\left(\int_a^b\tilde{T}_{n-2}(s,\tau)f^{(n)}(\tau)d\tau\right)ds$$

and then using Fubini's theorem in last term to get (2.10).

Also, if we replace f by f'' and n by $n-2$ ($n\geq 3$) in formula (1.7) and rearranging indices we get

$$f''(s)=\frac{f'(a)-f'(b)}{b-a}+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}\\ +\frac{1}{(n-3)!}\int_a^b T_{n-2}(s,\tau)f^{(n)}(\tau)d\tau. \tag{2.15}$$

Similarly, using (2.15) in (2.13) and applying Fubini's Theorem, we get (2.12). □

Theorem 2.3. *Let all assumptions of Theorem* 2.2 *be valid with additional condition*

$$\int_a^b\sum_{i\in I_m}\omega_i G_l(\eta_i,s)\tilde{T}_{n-2}(s,\tau)ds\geq 0,\quad \forall\,\tau\in[a,b]. \tag{2.16}$$

Then for every $n-$convex function $f: I \to \mathbb{R}$ following inequality is valid

$$\sum_{i \in I_m} \omega_i f(\eta_i) \geq \frac{f'(a)-f'(b)}{b-a} \int_a^b \sum_{i \in I_m} \omega_i G_l(\eta_i, s) ds$$
$$+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \sum_{i \in I_m} \omega_i G_l(\eta_i, s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \quad (2.17)$$

Proof. Since function f is $n-$convex, therefore $f^{(n)} \geq 0$. Hence using n convexity of function and (2.16) in (2.10) we obtain our required result. □

Theorem 2.4. *Let all assumptions of Theorem* 2.2 *be valid with additional condition*

$$\int_a^b \sum_{i \in I_m} \omega_i G_l(\eta_i, s) T_{n-2}(s, \tau) ds \geq 0, \quad \forall \tau \in [a,b]. \quad (2.18)$$

Then for every $n-$convex function $f: I \to \mathbb{R}$ following inequality is valid

$$\sum_{i \in I_m} \omega_i f(\eta_i) \geq \frac{f'(a)-f'(b)}{b-a} \int_a^b \sum_{i \in I_m} \omega_i G_l(\eta_i, s) ds$$
$$+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \sum_{i \in I_m} \omega_i G_l(\eta_i, s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \quad (2.19)$$

Proof. Since function f is $n-$convex, therefore $f^{(n)} \geq 0$. Hence using n convexity of function and (2.18) in (2.12) we obtain what we wanted. □

Now we state important consequences.

Theorem 2.5. *Let all assumptions from Theorem* 2.2 *be valid with*

$$\sum_{i \in I_m} \omega_i = 0 \quad \text{and} \quad \sum_{i \in I_m} \omega_i |\eta_i - \eta_k| \geq 0 \text{ for } k \in I_m. \quad (2.20)$$

If f is n-convex and n is even, then inequalities (2.17) *and* (2.19) *are valid.*

Proof. Since new Green functions $G_l(s,\tau)$ are convex with respect to t for all $s \in [a,b]$ and for each $l \in I_4$ and $\eta = (\eta_1, \ldots, \eta_m)$ and $\omega = (\omega_1, \ldots, \omega_m)$ satisfy conditions (1.4) from Remark 1.2, therefore, we have

$$\sum_{i \in I_m} \omega_i G_l(\eta_i, s) \geq 0 \quad \text{for} \quad s \in [a,b]. \quad (2.21)$$

Since $\tilde{T}_{n-2}(s,\tau)$ and $T_{n-2}(s,\tau)$ both are non-negative for even values of n, therefore combining present fact with (2.21) we get inequality (2.16) and inequality (2.18) respectively. As f is n-convex, so results follows from Theorem 2.3 and Theorem 2.4 respectively. □

Integral version may be stated as under:

Theorem 2.6. *Fix $l \in I_4$. Let $\lambda : [\alpha,\beta] \to [a,b]$ be a function and let $\omega : [\alpha,\beta] \to \mathbb{R}$ be a continuous integrable function such that $\int_\alpha^\beta \omega(\eta)d\eta = 0$ and $\int_\alpha^\beta \omega(\eta)\lambda(\eta)d\eta = 0$. Let $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)} \in AC(I)$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$, then for all $s \in [a,b]$ we have following identity*

$$\int_\alpha^\beta \omega(\eta) f(\lambda(\eta))\, d\eta = \frac{f'(a)-f'(b)}{b-a} \int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta\, ds$$
$$+\sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \left(\int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta \right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$+\frac{1}{(n-3)!} \int_a^b f^{(n)}(\tau) \left(\int_a^b \left(\int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta \right) \tilde{T}_{n-2}(s,\tau) ds \right) d\tau. \quad (2.22)$$

Moreover, we also have

$$\int_\alpha^\beta \omega(\eta) f(\lambda(\eta))\, d\eta = \frac{f'(a)-f'(b)}{b-a} \int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta\, ds$$
$$+\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \left(\int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta \right)$$
$$\times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$+\frac{1}{(n-3)!} \int_a^b f^{(n)}(\tau) \left(\int_a^b \left(\int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta \right) T_{n-2}(s,\tau) ds \right) d\tau \quad (2.23)$$

where $\tilde{T}_n$, T_n and G_l are as in Theorem 2.2.

Theorem 2.7. *Let all assumptions of Theorem 2.6 be valid with following condition*

$$\int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, \tilde{T}_{n-2}(s,\tau)\, d\eta\, ds \geq 0, \quad \forall\, \tau \in [a,b]. \quad (2.24)$$

Then for every $n-$convex function $f : I \to \mathbb{R}$ following inequality is valid

$$\int_\alpha^\beta \omega(\eta) f(\lambda(\eta))\, d\eta \geq \frac{f'(a)-f'(b)}{b-a} \int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta\, ds + \sum_{k=2}^{n-1} \frac{k}{(k-1)!}$$
$$\times \int_a^b \left(\int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta \right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \quad (2.25)$$

Theorem 2.8. *Let all assumptions of Theorem 2.6 be valid with following condition*

$$\int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, T_{n-2}(s,\tau) d\eta\, ds \geq 0, \quad \forall\, \tau \in [a,b]. \quad (2.26)$$

Then for all n−convex function $f : I \to \mathbb{R}$ following inequality is valid

$$\int_{\alpha}^{\beta} \omega(\eta) f(\lambda(\eta))\, d\eta \geq \frac{f'(a)-f'(b)}{b-a} \int_a^b \int_{\alpha}^{\beta} \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta\, ds + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \left(\int_{\alpha}^{\beta} \omega(\eta)\, G_l(\lambda(\eta),s)\, d\eta \right) \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \quad (2.27)$$

Theorem 2.9. *Let all assumptions from Theorem* 2.6 *be valid with additional assumptions that $\omega : [\alpha,\beta] \to \mathbb{R}$ and $\lambda : [\alpha,\beta] \to [a,b]$ be such that*

$$\begin{aligned} \int_{\alpha}^{\beta} \omega(\eta) \lambda(\eta)^k\, d\eta = 0, \quad & \forall\, k \in I_{n-1} \\ \int_{\alpha}^{\beta} \omega(\eta)\, (\lambda(\eta)-t)_+^{n-1}\, d\eta \geq 0, \quad & \forall\, t \in [a,b]. \end{aligned} \quad (2.28)$$

If f is n-convex and n is even, then inequalities (2.25) *and* (2.27) *are valid.*

2.1. **Inequalities for convex functions of order *n* at a point.** In articles [7] and [17], we can find following definition of convexity at a point.

Definition 2.10. Let I be an interval in $\mathbb{R}$, $\lambda \in I^{\circ}$ and n is a non-negitive integer. A function $f : I \to \mathbb{R}$ is called n-convex at point p if there exists a constant C such that function

$$F(\eta) = f(\eta) - \frac{C}{(n-1)!} \eta^{n-1}$$

is $(n-1)$-concave on $I \cap (-\infty, p]$ and $(n-1)$-convex on $I \cap [p, \infty)$.

Here we improve results from previous subsection.

Let $T_n^{[a,c]}$ and $T_n^{[c,b]}$ denote equivalent of (1.8) on these intervals, i. e.,

$$T_n^{[a,c]}(s,\tau) = \begin{cases} -\dfrac{(s-\tau)^n}{n(c-a)} + \dfrac{\eta - a}{c-a}(s-\tau)^{n-1}, & a \leq \tau \leq s \leq c, \\[2ex] -\dfrac{(s-\tau)^n}{n(c-a)} + \dfrac{\eta - c}{c-a}(s-\tau)^{n-1}, & a \leq s \leq \tau \leq c, \end{cases} \quad (2.29)$$

$$T_n^{[c,b]}(s,\tau) = \begin{cases} -\dfrac{(s-\tau)^n}{n(b-c)} + \dfrac{\eta - c}{b-c}(s-\tau)^{n-1}, & c \leq \tau \leq s \leq b, \\[2ex] -\dfrac{(s-\tau)^n}{n(b-c)} + \dfrac{\eta - b}{b-c}(s-\tau)^{n-1}, & c \leq s \leq \tau \leq b. \end{cases} \quad (2.30)$$

Similarly, $\tilde{T}_n^{[a,c]}$ and $\tilde{T}_n^{[c,b]}$ denote equivalent of (2.11) on these intervals, i. e.,

$$\tilde{T}_{n-2}^{[a,c]}(s,\tau)=\begin{cases}\frac{1}{c-a}\left[\frac{(s-\tau)^{n-2}}{(n-2)}+(s-a)(s-\tau)^{n-3}\right], & a\le\tau\le s\le c,\\ \frac{1}{c-a}\left[\frac{(s-\tau)^{n-2}}{(n-2)}+(s-c)(s-\tau)^{n-3}\right], & a\le s<\tau\le c,\end{cases} \tag{2.31}$$

$$\tilde{T}_{n-2}^{[c,b]}(s,\tau)=\begin{cases}\frac{1}{b-c}\left[\frac{(s-\tau)^{n-2}}{(n-2)}+(s-c)(s-\tau)^{n-3}\right], & c\le\tau\le s\le b,\\ \frac{1}{b-c}\left[\frac{(s-\tau)^{n-2}}{(n-2)}+(s-b)(s-\tau)^{n-3}\right], & c\le s<\tau\le b.\end{cases} \tag{2.32}$$

Let $\eta\in[a,c]^{n_1}$, $\omega\in\mathbb{R}^{n_1}$, $y\in[c,b]^{n_2}$ and $q\in\mathbb{R}^{n_2}$ and denote

$$A_1(f,[a,c])=\sum_{i\in I_{n_1}}\omega_i f(\eta_i)-\frac{f'(a)-f'(c)}{c-a}\int_a^c\sum_{i\in I_{n_1}}\omega_i G_l(\eta_i,s)ds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^c\sum_{i\in I_{n_1}}\omega_i G_l(\eta_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(c)(s-c)^{k-1}}{c-a}ds, \tag{2.33}$$

$$A_1(f,[c,b])=\sum_{i\in I_{n_2}}q_i f(y_i)-\frac{f'(c)-f'(b)}{c-b}\int_c^b\sum_{i\in I_{n_2}}q_i G_l(y_i,s)ds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_c^b\sum_{i\in I_{n_2}}q_i G_l(y_i,s)\frac{f^{(k)}(c)(s-c)^{k-1}-f^{(k)}(c)(s-b)^{k-1}}{b-c}ds. \tag{2.34}$$

Using (2.10) we may also define

$$A_1(f,[a,c])=\frac{1}{(n-3)!}\int_a^c f^{(n)}(\tau)\left(\int_a^c\sum_{i\in I_{n_1}}\omega_i G_l(\eta_i,s)\tilde{T}_{n-2}^{[a,c]}(s,\tau)ds\right)d\tau, \tag{2.35}$$

$$A_1(f,[c,b])=\frac{1}{(n-3)!}\int_c^b f^{(n)}(\tau)\left(\int_c^b\sum_{i\in I_{n_2}}q_i G_l(y_i,s)\tilde{T}_{n-2}^{[c,b]}(s,\tau)ds\right)d\tau. \tag{2.36}$$

In same manner we can introduce further functionals namely

$$A_2(f,[a,c])=\frac{1}{(n-3)!}\int_a^c f^{(n)}(\tau)\left(\int_a^c\sum_{i\in I_{n_1}}\omega_i G_l(\eta_i,s)T_{n-2}^{[a,c]}(s,\tau)ds\right)d\tau, \tag{2.37}$$

$$A_2(f,[c,b])=\frac{1}{(n-3)!}\int_c^b f^{(n)}(\tau)\left(\int_c^b\sum_{i\in I_{n_2}}q_i G_l(y_i,s)T_{n-2}^{[c,b]}(s,\tau)ds\right)d\tau. \tag{2.38}$$

$$A_3(f,[a,c]) = \frac{1}{(n-3)!}\int_a^c f^{(n)}(\tau)\left(\int_a^c\left(\int_\alpha^\beta \omega(\eta)\,G_l(\lambda(\eta),s)\,d\eta\right)\tilde{T}_{n-2}^{[a,c]}(s,\tau)ds\right)d\tau, \tag{2.39}$$

$$A_3(f,[c,b]) = \frac{1}{(n-3)!}\int_c^b f^{(n)}(\tau)\left(\int_c^b\left(\int_\gamma^\delta q(\eta)\,G_l(h(\eta),s)\,d\eta\right)\tilde{T}_{n-2}^{[c,b]}(s,\tau)ds\right)d\tau, \tag{2.40}$$

$$A_4(f,[a,c]) = \frac{1}{(n-3)!}\int_a^c f^{(n)}(\tau)\left(\int_a^c\left(\int_\alpha^\beta \omega(\eta)\,G_l(\lambda(\eta),s)\,d\eta\right)T_{n-2}^{[a,c]}(s,\tau)ds\right)d\tau, \tag{2.41}$$

$$A_4(f,[c,b]) = \frac{1}{(n-3)!}\int_c^b f^{(n)}(\tau)\left(\int_c^b\left(\int_\gamma^\delta q(\eta)\,G_l(h(\eta),s)\,d\eta\right)T_{n-2}^{[c,b]}(s,\tau)ds\right)d\tau, \tag{2.42}$$

where $\alpha \le \beta, \gamma \le \delta$, $a < c < b$, $\lambda : [\alpha,\beta] \to [a,c]$, $\omega : [\alpha,\beta] \to \mathbb{R}$, $h : [\gamma,\delta] \to [c,b]$, $q : [\gamma,\delta] \to \mathbb{R}$ are integrable functions.

Theorem 2.11. *Let $\eta \in [a,c]^{n_1}$, $\omega \in \mathbb{R}^{n_1}$, $y \in [c,b]^{n_2}$ and $q \in \mathbb{R}^{n_2}$ be such that*

$$\int_a^c \sum_{i\in I_{n_1}} \omega_i G_l(\eta_i,s)\tilde{T}_{n-2}^{[a,c]}(s,\tau)\,ds \ge 0, \quad \forall\ \tau \in [a,c], \tag{2.43}$$

$$\int_c^b \sum_{i\in I_{n_2}} q_i G_l(y_i,s)\tilde{T}_{n-2}^{[c,b]}(s,\tau)\,ds \ge 0, \quad \forall\ \tau \in [c,b], \tag{2.44}$$

$$\int_a^c\int_a^c \sum_{i\in I_{n_1}} \omega_i G_l(\eta_i,s)\tilde{T}_{n-2}^{[a,c]}(s,\tau)\,dsd\tau = \int_c^b\int_c^b \sum_{i\in I_{n_2}} q_i G_l(y_i,s)\tilde{T}_{n-2}^{[c,b]}(s,\tau)\,dsd\tau, \tag{2.45}$$

where $\tilde{T}_n^{[a,c]}$, $\tilde{T}_n^{[c,b]}$, $A_1(\cdot,[a,c])$ and $A_1(\cdot,[c,b])$ are given by (2.31), (2.32), (2.33) *and* (2.34) *respectively. If $f : [a,b] \to \mathbb{R}$ is $(n+1)-$convex at point c, then*

$$A_1(f,[a,c]) \le A_1(f,[c,b]). \tag{2.46}$$

If inequalities in (2.43) *and* (2.44) *are reversed, then* (2.46) *is valid with reversed sign of inequality.*

Proof. For sake of brevity we let $A(\cdot) = A_1(\cdot,[a,b])$ and $B(\cdot) = A_1(\cdot,[c,b])$. Let $F = f - \frac{C}{n!}e_n$ be as in definition of function n-convex at a point, i. e., function F is n-concave on $[a,c]$ and n-convex on $[c,b]$ and e_i denote monomials $e_i(\eta) = \eta^i$, $i \in \mathbb{N}\cup\{0\}$. Applying Theorem 2.3 to F on interval $[a,c]$ we have

$$0 \ge A(F) = A(f) - \frac{C}{n!}A(e_n) \tag{2.47}$$

and applying Theorem 2.3 to F on interval $[c,b]$ we have

$$0 \leq B(F) = B(f) - \frac{C}{n!} B(e_n). \tag{2.48}$$

Identities (2.35) and (2.36) applied to function e_n yield

$$\begin{aligned} A(e_n) &= \frac{n!}{(n-3)!} \int_a^c \left(\sum_{i \in I_m} \omega_i \tilde{T}_{n-2}^{[a,c]}(\eta_i, s) \right) ds, \\ B(e_n) &= \frac{n!}{(n-3)!} \int_c^b \left(\sum_{i=1}^{l} q_i \tilde{T}_{n-2}^{[c,b]}(y_i, s) \right) ds. \end{aligned}$$

Therefore, assumption (2.45) is equivalent to $A(e_n) = B(e_n)$. Now, from (2.47) and (2.48) we obtain stated inequality. □

Remark 2.12. From proof of Theorem 2.11 we have

$$A_1(f,[a,c]) \leq \frac{C}{n!} A_1(e_n,[a,c]) = \frac{C}{n!} A_1(e_n,[c,b]) \leq A_1(f,[c,b]).$$

In fact, inequality (2.46) still is valid if we replace assumption (2.45) with weaker assumption that

$$C\left(A_1(e_n,[c,b]) - A_1(e_n,[a,c])\right) \geq 0.$$

Here we have another similar result.

Theorem 2.13. *Let $\eta \in [a,c]^{n_1}$, $\omega \in \mathbb{R}^{n_1}$, $y \in [c,b]^{n_2}$ and $q \in \mathbb{R}^{n_2}$ be such that*

$$\int_a^c \sum_{i \in I_{n_1}} \omega_i G_l(\eta_i, s) T_{n-2}^{[a,c]}(s,\tau)\, ds \geq 0, \quad \forall\, \tau \in [a,c], \tag{2.49}$$

$$\int_c^b \sum_{i \in I_{n_2}} q_i G_l(y_i, s) T_{n-2}^{[c,b]}(s,\tau)\, ds \geq 0, \quad \forall\, \tau \in [c,b], \tag{2.50}$$

$$\int_a^c \int_a^c \sum_{i \in I_{n_1}} \omega_i G_l(\eta_i, s) T_{n-2}^{[a,c]}(s,\tau)\, ds d\tau = \int_c^b \int_c^b \sum_{i \in I_{n_2}} q_i G_l(y_i, s) T_{n-2}^{[c,b]}(s,\tau)\, ds d\tau, \tag{2.51}$$

where $T_n^{[a,c]}$, $T_n^{[c,b]}$, $A_2(f,[a,c])$ and $A_2(f,[c,b])$ are given by (2.29), (2.30), (2.37) *and* (2.38) *respectively. If $f : [a,b] \to \mathbb{R}$ is $(n+1)$-convex at point c, then*

$$A_2(f,[a,c]) \leq A_2(f,[c,b]). \tag{2.52}$$

If inequalities in (2.49) *and* (2.50) *are reversed, then* (2.52) *is valid with reversed sign of inequality.*

Remark 2.14. Similar results can also be stated for integral versions as well by using functionals $A_3(\cdot,[a,c]), A_3(\cdot,[c,b]), A_4(\cdot,[a,c])$ and $A_4(\cdot,[c,b])$ as defined in (2.39), (2.40) (2.41) and (2.42) respectively.

3. Bounds for identities related to generalized linear inequalities

Let $f_1, f_2 \in L[a,b]$. We write Čebyšev functional

$$T(f_1,f_2) = \frac{1}{b-a}\int_a^b f_1(\eta)f_2(\eta)d\eta - \left(\frac{1}{b-a}\int_a^b f_1(\eta)d\eta\right)\left(\frac{1}{b-a}\int_a^b f_2(\eta)d\eta\right). \tag{3.1}$$

Here we give several estimations connected with functionals $A_k(\cdot,[\cdot,\cdot])$, $k \in I_4$ defined under assumptions of Theorems 2.3, 2.4, 2.7 and 2.8 respectively on interval $[a,b]$ as defined in (2.35), for example,

$$A_1(f,[a,b]) = \frac{1}{(n-3)!}\int_a^b f^{(n)}(\tau)\left(\int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)\tilde{T}_{n-2}^{[a,b]}(s,\tau)ds\right)d\tau.$$

For main results of present section we need some notation to work efficiently. Under assumptions of Theorems 2.3, 2.4, 2.7 and 2.8 respectively, we introduce following notations

$$\Omega_1(\tau) = \int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)\tilde{T}_{n-2}(s,\tau)ds, \quad \forall\, \tau \in [a,b], \tag{3.2}$$

$$\Omega_2(\tau) = \int_a^b \sum_{i\in I_m} \omega_i G_l(\eta_i,s)T_{n-2}(s,\tau)ds, \quad \forall\, \tau \in [a,b], \tag{3.3}$$

$$\Omega_3(\tau) = \int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, \tilde{T}_{n-2}(s,\tau)\, d\eta\, ds, \quad \forall\, \tau \in [a,b], \tag{3.4}$$

$$\Omega_4(\tau) = \int_a^b \int_\alpha^\beta \omega(\eta)\, G_l(\lambda(\eta),s)\, T_{n-2}(s,\tau)\, d\eta\, ds, \quad \forall\, \tau \in [a,b]. \tag{3.5}$$

Hence Čebyšev functional is written as for Ω:

$$T(\Omega,\Omega) = \frac{1}{b-a}\int_a^b \Omega^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b \Omega(s)\,ds\right)^2.$$

A bound for Čebyšev functional is given in following proposition in which pre-Grüss inequality is given (see [12]).

Proposition 3.1. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be integrable such that $f_1f_2 \in L(a,b)$. If*

$$\gamma_1 \le f_2(\eta) \le \gamma_2 \quad \textit{for} \quad \eta \in [a,b],$$

then

$$|T(f_1,f_2)| \le \frac{1}{2}(\gamma_2-\gamma_1)\sqrt{T(f_1,f_1)},$$

where, γ_1, γ_2 are real constants.

The following results can be found in [4]:

Proposition 3.2. *Let $f_1 \in L[a,b]$ and let $f_2 \in AC[a,b]$ with $(\cdot - a)(b - \cdot)[f_2']^2 \in L[a,b]$. Then we have inequality*

$$|T(f_1,f_2)| \leq \frac{1}{\sqrt{2}} \left(\frac{1}{b-a} |T(f_1,f_1)| \int_a^b (\eta - a)(b-\eta)[f_2'(\eta)]^2 d\eta \right)^{1/2}. \tag{3.6}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.6) is best possible.

Proposition 3.3. *Let $f_2 : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function and let $f_1 \in AC[a,b]$ such that $f_1' \in L_\infty[a,b]$. Then we have inequality*

$$|T(f_1,f_2)| \leq \frac{1}{2(b-c)} \|f_1'\|_\infty \int_a^b (\eta - a)(b-\eta) df_2(\eta). \tag{3.7}$$

The constant $\frac{1}{2}$ in (3.7) is best possible.

At present stage we denote $A(\cdot,[a,b])$ by simply $A_k(\cdot)$ for $k \in I_4$.

Theorem 3.4. *Let $k \in I_4$. Let $f : I \to \mathbb{R}$, $[a,b] \subset I$, be such that $f^{(n-1)} \in AC(I)$ and*

$$\gamma \leq f^{(n)}(\eta) \leq \Gamma \quad \text{for } \eta \in [a,b].$$

Then in representation

$$A_k(f) = \frac{\left[f^{(n-1)}(b) - f^{(n-1)}(a) \right]}{(n-3)!(b-a)} \int_a^b \Omega_k(s) ds + R_n^k, \tag{3.8}$$

remainder R_n^k satisfies estimation

$$|R_n^k| \leq \frac{b-a}{2(n-3)!} (\Gamma - \gamma) \sqrt{T(\Omega_k, \Omega_k)}. \tag{3.9}$$

Proof. Fix $k \in I_4$. Using definition of A_k and results from previous subsection we have

$$\begin{aligned} A_k(f) &= \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \Omega_k(s) ds \\ &= \frac{1}{(n-3)!(b-a)} \int_a^b f^{(n)}(s) ds \int_a^b \Omega_k(s) ds + R_n^k \\ &= \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b \Omega_k(s) ds + R_n^k, \end{aligned}$$

where

$$R_n^k = \frac{1}{(n-3)!} \left(\int_a^b f^{(n)}(s) \Omega_k(s) ds - \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \int_a^b \Omega_k(s) ds \right).$$

Applying Proposition 3.1 for $f_1 \to \Omega_k$ and $f_2 \to f^{(n)}$, we obtain

$$|R_n^k| = \frac{1}{(n-3)!} |T(\Omega_k, f^{(n)}| \leq \frac{b-a}{2(n-3)!} (\Gamma - \gamma) \sqrt{T(\Omega_k, \Omega_k)}.$$

□

Now we state some Ostrowski-type inequalities related to our established inequalities.

Theorem 3.5. *Let $k \in I_4$. Let (q,r) be such that $1 \le q, r \le \infty$, $\frac{1}{q}+\frac{1}{r}=1$. Let $f^{(n)} \in L_q[a,b]$ for some $n \ge 3$. Then*

$$|A_k(f)| \le \frac{1}{(n-3)!} \|f^{(n)}\|_q \, \|\Omega_k\|_r . \tag{3.10}$$

The constant on RH side of (3.10) *is sharp for $1 < q \le \infty$ and best possible for $q = 1$.*

Proof. Fix $k \in I_4$. From definition of A_k and results from second section, applying Hölder inequality we get

$$|A_k(f)| = \left| \frac{1}{(n-3)!} \int_a^b f^{(n)}(s)\Omega_k(s)ds \right| \le \|f^{(n)}\|_q \, \|\lambda_k\|_r .$$

Let us denote a quotient $\frac{1}{(n-3)!}\Omega_k$ by λ_k. For proof of sharpness of

$$\left(\int_a^b |\lambda_k(\tau)|^r \, d\tau \right)^{1/r},$$

here we let function f for which equality in (3.10) is valid.

For $1 < q < \infty$ take f to be such that

$$f^{(n)}(\tau) = \operatorname{sgn} \lambda_k(\tau) \cdot |\lambda_k(\tau)|^{1/(q-1)}.$$

For $q = \infty$, take f such that

$$f^{(n)}(\tau) = \operatorname{sgn} \lambda_k(\tau).$$

The fact that (3.10) is best possible for $q = 1$ can be proved as in Theorem 5.18 of [10]. □

4. Mean Value Theorems

Now we state MVT of Cauchy- and Lagrange- type for A_k, $k \in I_4$. Here $f_0(\eta) = \frac{\eta^n}{n!}$.

Theorem 4.1. *Let $f \in C^n[a,b]$ and let $A_k : C^n[a,b] \to \mathbb{R}$ for $k \in I_4$ be linear functionals as defined earlier. Then there exist $\xi_k \in [a,b]$ for $k \in I_4$ such that*

$$A_k(f) = f^{(n)}(\xi_k) A_k(f_0). \tag{4.1}$$

Proof. Since $f^{(n)}$ is continuous on $[a,b]$, so $L \le f^{(n)}(\eta) \le U$ for $\eta \in [a,b]$, where $L = \min_{\eta \in [a,b]} f^{(n)}(\eta)$ and $U = \max_{\eta \in [a,b]} f^{(n)}(\eta)$.

Therefore function

$$F(\eta) = U\frac{\eta^n}{n!} - f(\eta) = U f_0(\eta) - f(\eta)$$

produces

$$F^{(n)}(\eta) = U - f^{(n)}(\eta) \ge 0,$$

i.e., F is $n-$convex. Hence $A_k(F) \geq 0$ and we conclude that for $k \in I_4$

$$A_k(f) \leq UA_k(f_0).$$

Similarly, for $k \in I_4$ we have

$$LA_k(f_0) \leq A_k(f).$$

Combining two inequalities we get

$$LA_k(f_0) \leq A_k(f) \leq UA_k(f_0)$$

and we easily arrive at (4.1). □

Theorem 4.2. *Let $f, N \in C^n[a,b]$ and let $A_k : C^n[a,b] \to \mathbb{R}$ for $k \in I_4$ be linear functionals as defined earlier. Then there exist $\xi_k \in [a,b]$ for $k \in I_4$ such that*

$$\frac{A_k(f)}{A_k(N)} = \frac{f^{(n)}(\xi_k)}{N^{(n)}(\xi_k)}$$

assuming non-zero denominators.

Proof. Fix $k \in I_4$. Let $\omega \in C^n[a,b]$ be defined as

$$\omega = A_k(N)f - A_k(f)N.$$

Using Theorem 4.1 there exist ξ_k such that

$$0 = A_k(\omega) = \omega^{(n)}(\xi_k)A_k(f_0)$$

or

$$[A_k(N)f^{(n)}(\xi_k) - A_k(f)N^{(n)}(\xi_k)]\Lambda_k(f_0) = 0$$

which gives us required result. □

Remark 4.3. If inverse of $\frac{f^{(n)}}{N^{(n)}}$ exists, then from previous mean value results we may state generalized means as

$$\xi_k = \left(\frac{f^{(n)}}{N^{(n)}}\right)^{-1}\left(\frac{A_k(f)}{A_k(N)}\right), \quad k \in I_4. \tag{4.2}$$

5. CONCLUSION

In year 2020 an article published in Arabian Journal of Mathematics with the title "Weighted averages of $n-$convex functions via extension of Montgomery identity" in which authors stated results related to linear inequalities via extension of Montgomery identity and Green function involving convex functions of order n [8]. In this article using similar techniques we have proved similar results for four new Green functions which were first introduced in the article [3]. The new thing in this article is that we have studied $n-$convexity at a point for our proposed inequalities. Other new things in this article are the bounds of reminders for our proposed results using pre-Grüss and other related inequalities; such results were not previously studied in the article [8].

Competing interests

The authors declare that they have no competing interests.

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Author's contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

References

[1] Aljinović, A. A., Pečarić, J., Vukelić A. (2005). *On some Ostrowski type inequalities via Montgomery identity and Taylor's formula II*, Tamkang Jour. Math. **36** (4), 279-301.

[2] Bernstein, S. N. (1929). *Sur les fonctions absolument monotones*, *Acta Math.* **52** (1), 1–66.

[3] Butt, S. I., Mehmood, N., Pečarić, J. (2017), New generalizations of Popoviciu type inequalities via new Green functions and Fink's identity, Transactions of A. Razmadze Mathematical Institute, **171** (3), 293-303.

[4] Cerone, P., Dragomir, S. S. (2014). *Some new Owstrowski-type bounds for Čebyšev functional and applications*, J. Math. Inequal., **8**(1), 159-170.

[5] Jakšetić, J., Pečarić, J. (2013). *Exponential convexity method*, J. Convex Anal., **20** (1), 181–197.

[6] Khan, A. R., Pečarić, J. E., Varošanec, S. (2013). *Popoviciu type characterization of positivity of sums and integrals for convex functions of higher order*, J. Math. Inequal., **7** (2), 195–212.

[7] Khan, A. R., Pečarić, J. E., Praljak, M. (2016). *Popoviciu type inequalities for n convex functions via extension of Montgomery identity*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **24** (3), 161–188

[8] Khan, A. R., Pečarić, J. E., Praljak, M. (2020). *Weighted averages of* $n-$*convex functions via extension of Montgomery identity*, Arab. J. Math, **2020** (9), 381–392.

[9] Khan, A. R., Latif, N., Pečarić, J. E. (2012). *Exponential convexity for majorization*, J. Inequal. Appl., **2012**, atr 105, 1–13.

[10] Khan, A. R., Pečarić, J. E., Praljak, M., Varošanec, S. (2017). *General Linear Inequalities and Positivity / Higher order convexity*. Monographs in inequalities 12, Element, Zagreb, pp. 269.

[11] Marshall, A. W., Olkin, I., Arnold, B. C. (2011). *Inequalities: theory of majorization and its applications*, 2nd Ed., Springer, New York.

[12] Matić, M., Pečarić, J., Ujević, N. (2000). *Improvement and further generalization of some inequalities of Ostrowski-Grüss type*, Computer Math. Appl. **39**(3-4), 161–175.

[13] Mitrinović, D. S., Pečarić, J. E., Fink, A. M. (1994). *Inequalities for functions and their Integrals and Derivatives, Kluwer* Academic Publishers, Dordrecht.

[14] Pečarić, J. (1991). *On Jessens Inequality for Convex Functions, III*, J. Math. Anal. Appl., **156**, 231-239.

[15] Pečarić, J., Perić J. (2012). *Improvements of Giaccardi and Petrović inequality and related Stolarsky type means,* An. Univ. Craiova Ser. Mat. Inform., **39** (1), 65–75.

[16] Pečarić, J., Perušić, A., Smoljak, K. (2015). *Generalizations of Steffensens Inequality by Abel-Gontscharoff Polynomial*, Khayyam J. Math. **1** (1), 45-61.

[17] Pečarić, J., Praljak, M., Witkowski A. (2015). *Linear operator inequality for convex functions of order n at a point*, Math. Ineq. Appl. **18** (4), 1201–1217.

[18] Pečarić, J. E., Proschan, F., Tong, Y. L. (1992). *Convex functions, partial orderings and statistical applications* Academic Press, New York.

[19] Popoviciu,T. (1940). *Notes sur les fonctions convexes d'orde superieur III,* Mathematica (Cluj) **16**, 74-86.

[20] Popoviciu,T.(1940). *Notes sur les fonctions convexes d'orde superieur IV,* Disquisitiones Math. **1**, 163-171.

[21] Popoviciu,T.(1941). *Notes sur les fonctions convexes d'orde superieur IX,* Bull. Math. Soc. Roumaine Sci. **43**, 85–141.

[22] Popoviciu,T. (1944). *Les fonctions convexes*, Herman and Cie, Editeurs, Paris.

[23] Widder, D. V. (1942). *Completely convex function and Lidstone series*, Trans. Am. Math. Soc. **51**, 387–398.

[24] Widder, D. V. (1934). *Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral,* Bull. Amer. Math. Soc., **40** (4), 321–326.