ISSN 2709-4170 (Print)
https://doi.org/10.52587/TMCS0102061
Transactions in Mathematical and Computational Sciences
Vol. 1(2)(2021) pp. 68–75

CONSTRUCTION OF CATMULL-CLARK SUBDIVISION SCHEME

SITI ZUHAIRAH ZAINI AZLAN[1], MD YUSHALIFY MISRO[1,*]

[1]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Pulau Pinang, Malaysia

Abstract. Subdivision surface is a versatile tool for representing a smooth surface with any topology. This research explains how a smooth polyhedron surface is made using the Catmull-Clark subdivision method. The approach is based on the consideration of the regular topological entities of polyhedron on a cube. Construction is seen as a generalization of an arbitrary control point mesh recurrent subdivision algorithm. For faces, edges and arbitrary net points, the process uses the same expression that is formed in the cube. The process of the scheme will produce a smooth surface as the result. The important criteria for the construction also presented.
Keywords: Subdivision, Catmull-Clark, Surface.

1. Introduction

Recently, the local animation industry has evolved over the past three decades. Catmull-Clark subdivision surfaces are a widely used technique in 3D computer graphics to construct complicated geometries. In [1], the researcher proposed to combine Catmull-Clark subdivision surfaces with Isogeometric boundary element method to simulate acoustic propagation in semi-infinite domains. The application of Isogeometric Analysis based on extended Catmull-Clark subdivision approach for the minimal surface models on planar domain has been proposed in [2]. [3] construct a new control lattice, whose limit volume by the Catmull-Clark subdivision scheme that interpolates vertices of the original hexahedral mesh. In 1974, Chaikin presented the concept of generating a polygon curve by refining the polygon successively by introducing new vertices and edges. In [4], subdivision surfaces were discovered simultaneously by two pairs of researchers namely, Edwin Catmull and Jim Clark as well as Daniel Doo and Malcolm Sabin.

[4] presented a method to recurrently generate surfaces that approximate the arbitrary topology mesh points. The method was developed as an extension of the recursive bicubic B-spline patch subdivision algorithm. An exceptional vertex is a vertex with a valence of under four of the initial mesh [5]. The number of edges at the original mesh was specified by this value. Even though the method proposed by Stam in [5] allows us treat dividing surfaces as parametric surfaces, [6] finds that certain computing robustness problems still occur around extraordinary points. Thus, the Stam approach has been altered such that the extraordinary point is stable as well as a normal vector convergence. The analysis of his first importance and his own subdivision matrix have been the basis for a condensed version of the assessment formula.

*Correspondence: yushalify@usm.my
Received: 17.03.2021 Revised: 16.06.2021 Accepted: 26.11.2021 Published: 31.12.2021

Based on Catmull-Clark base function precomputation, [7] has used an effective subdivision method that is suitable for parallelization at both single instruction level multi-data (SIMD) operations and parallel operating units. The maximum number of subdivision levels is selected for five levels. The objective was to eliminate memory delays and use the CPU cache. [8] introduced a description for neighboring points by using first order difference of control points of Catmull-Clark surfaces. Then, the rate of convergence of control meshes of Catmull-Clark surfaces will be obtained. By using the convergence results, it developed a computational formula for Catmull-Clark surfaces of division depth.

Subdivision surfaces also used in the making character animation such as Geris Game produced by Pixar Animation Studios. Tony DeRose is a senior scientist and the chief of Pixar Animation Studios' Research Department has developed subdivision surfaces in order to satisfy demands for high end output, so as to produce the credible and enthusiastic animation for characters in computer graphics. Subdivisions algorithm have been expanded by [9] to eliminate the limitations imposed by non-uniform B-splines (NURBS). NURBS trimmed surfaces has been introduced which can be transformed into Bézier edge-contained Catmull-Clark surfaces [10]. The surface of the NURBS and its curve is used as an input. The ultimate purpose of their work is to provide tools for both Catmull-Clark subdivision and CAD applications that need accuracy due to NURBS's strict rectangular topologies.

Doo and Sabin also created the second subdivision scheme in 1978. Restricted surfaces of Catmull-Clark and Doo-Sabin can be evaluated directly without recurrent refinement as mentioned in [5]. However, as a Doo-Sabin subdivision matrices are not necessarily diagonalizable, the solution is not as efficient computationally as the Catmull-Clark subdivision. In addition, the Catmull-Clark algorithm can also be determined easily. Thus, a subdivision scheme of Catmull-Clark was chosen to create a smooth polyhedron surface. Surface refining subdivisions can be widely divided into two groups. Interpolating schemes must fit the original vertices positions in the original mesh. In the meantime, approximating arrangements change the location of vertices required in the original mesh. Approximate systems are generally smoother, users are less controlled over the outcome.

This study present the concept of subdivision algorithm and to determine the number of vertices and faces for different type of polyhedrons during each iterative procedure. As we are familiar the subdivision algorithm plays a vital role in CAGD/CAD and animation industry. We can construct various high-end surfaces, especially in computer graphics to produce smooth surfaces by using the subdivision method. The relationship between faces and edges is very important especially when they need to analyze the surface using isogeometric analysis and finite element method as we need to calculate the number of elements after the refinement process.

This research will be organized as follows. Section 1 concerns about the implementation of subdivision surfaces and refinement schemes. Section 2 focuses on the methodology used to carry out this project, the Catmull-Clark subdivision scheme. Section 3 explores the significance of the construction of a smooth surface under the Catmull-Clark subdivision scheme. Finally, Section 4 sets out the conclusion of this project and the guidelines for future work.

2. METHODOLOGY

The polyhedron has three major topological entities, namely the faces, edges and vertices as in Figure 1. It is a solid object whose surface is made up of a number of flat faces bordered by straight lines. In fact, each face is a polygon, a closed form in a flat two-dimensional plane made up of points connected by straight lines. Polygons are not permitted to have holes inside them. A polygon is called regular if all of its sides are the same length, and all the angles between them are the same. A polyhedron is when the dimension is three-dimensional. It's a closed, solid object with a number of polygonal faces on its surface.

- Vertex: A vertex is a corner that represents a point that specifies where edges meet.
- Edge: An edge represents a finite line segment between faces.
- Face: A face represents a single flat surface of the polygon.

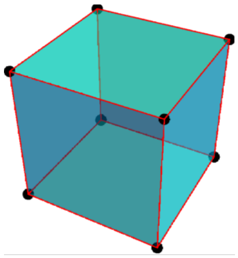


FIGURE 1. Regular polyhedron with vertex (black dot), edge (red) and faces (blue)

2.1. **Subdivision Surface.** The subdivision surface theory came up with the concept of constructing smooth free-form surfaces by iteratively refining coarse control grids. Step-by-step refinement is accomplished by repeatedly applying subdivision schemes to the developing grid. This process will produce finer control grids that converge to a smooth surface known as the limit surface. At each subdivision step, the original control mesh points (coordinates) are refined by adding new vertices, faces, and edges. As the number of subdivision stage approaches infinity, the control mesh converges to a limit surface. The limit surface can be shown to be continuous with a continuous tangent plane by carefully selecting the systems which add the new vertices, edges and faces.

A control polyhedron, which approximates the final surface, is the starting point for a subdivision surface. At each subdivision level, a new polyhedron is created by subdividing faces into smaller units that more closely resemble the final surface than the original faces. As a result, the polyhedron gets closer to the final surface.

2.2. **Catmull-Clark Subdivision.** The control vertices of the refined meshes are generated by a portfolio of weight coefficients from the control vertices of the previous stage. Finally, a limit surface made up of quadrilateral surface patches emerges from this set of meshes. The method used in this study is based on standard topological entities of polyhedrons on a cube. The cube is governed by 8 coordinates (black dot), as shown in Figure 2. This cube can be subdivided and required 26 sub-coordinates points as marked by blue dots. In the middles of the original mesh's squares, some of the blue dots merge with red dots, forming new face points. Similarly, these new dots are called new edge dots as they are placed at the edges of the original control points. New vertex points are the points that correspond to the old control points. It was discovered that each new control point of a given form was computed using the same algebraic expression as its neighbors when the original cube was split. For example, new face points are calculated as the average of the previous four vertices that face represented. This study describes the implementation of subdivision scheme to control-point meshes. Faces, edges, and points of arbitrary meshes are treated with the same expression that is generated in the cube. New face points are then calculated as the average old face points and new vertex points are calculated on the basis of the number of incident corners on top of a vertex, which results in the proper expression of a number 4, as in the cube. Now, the new front point labeled q is $(1,1)$, as in Figure 2. It gives

$$q_{11} = \frac{p_{11} + p_{12} + p_{21} + p_{22}}{4}. \tag{2.1}$$

Similarly, the point q_2, a new edge point, can be found by

$$q_2 = \frac{\frac{C+D}{2} + \frac{p_{12}+p_{22}}{2}}{2} \tag{2.2}$$

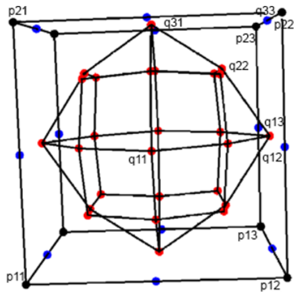


FIGURE 2. Cube with 8 control points

where,

$$q_{11} = C = \frac{p_{11} + p_{12} + p_{21} + p_{22}}{4}$$

and

$$q_{13} = D = \frac{p_{12} + p_{13} + p_{22} + p_{23}}{4}.$$

The new vertex point, q_{22}, is given by

$$q_{22} = \frac{Q}{4} + \frac{R}{2} + \frac{p_{22}}{4} \tag{2.3}$$

where,

$$Q = \frac{q_{11} + q_{13} + q_{31} + q_{33}}{4}$$

and

$$R = \frac{\frac{q_{11}+q_{12}}{2} + \frac{q_{12}+p_{22}}{2} + \frac{p_{22}+q_{31}}{2} + \frac{p_{31}+q_{11}}{2}}{4}.$$

Equations (1), (2), and (3) are simple to express to an arbitrary topology as a series of schemes which rely on located face points and the edges of a vertex. When the number is four, the schemes must produce Equations (1), (2), and (3). The scheme began with an arbitrary polyhedron mesh. The initial points are all vertices in the mesh. The new locations of the points work as follows when the surface is free of holes. The process of the scheme are shown as in Figure 3. To begin, face points for each face will be define, which is the average of all the original points representing the polyhedron's face. Next, the new vertex point will be locate, then the coordinate from the old points will be updated as in Equation (4)

$$R = \frac{Q + 2R + S(n-3)}{n} \tag{2.4}$$

where,

- Q= the average of the new face points
- R= the average of the midpoints of all old edge incident on the original vertex point
- S= old vertex point
- n= number of faces

Equation (4) is the barycenter of S, R and Q with the respective weight $(n-3), 2$ and 1. After calculation of these points, new edges are formed by

- each new face point is connected to the new edge points of the edges that define the old face.
- each new vertex point is connected to the new edge points of all old edge incident on the old vertex point.

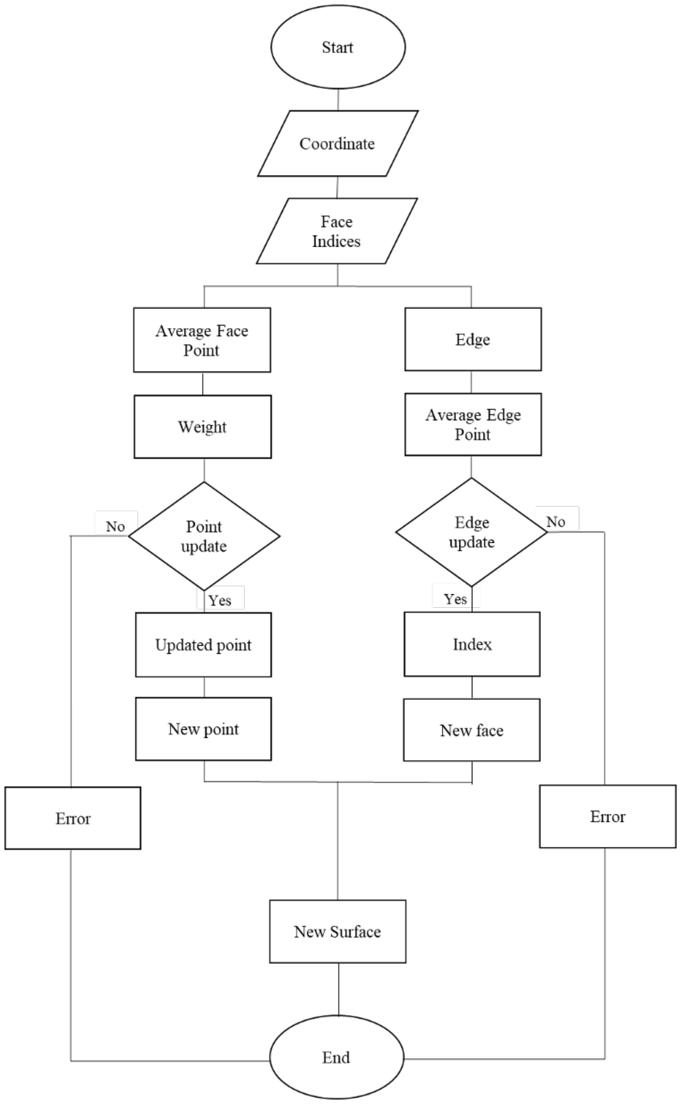


FIGURE 3. Flowchart of the scheme

Then every face with the new points is replaced by new faces. New faces are then identified as new edges surrounding them. The resulting surface will still be a quadrilateral mesh, which will not be planar in most situations. The new mesh would be finer in general than the old mesh.

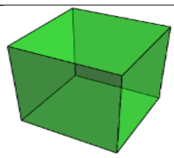
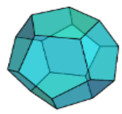
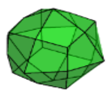
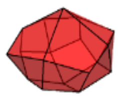
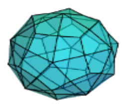
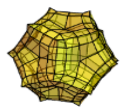
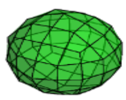
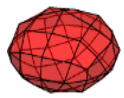
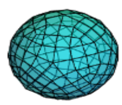
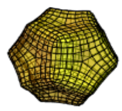
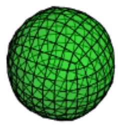
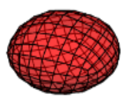
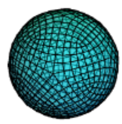
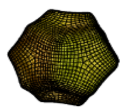
Smoother meshes are achieved by repeatedly subdividing the scheme. Although Catmull and Clark go to considerable lengths to show that the process converges to bicubic B-spline surfaces, they opt for the arbitrary-looking barycenter formula over a mathematical derivation in [4]. In case of point and edge fail to update, an error will occur and subdivision of the polyhedron failed to happen.

3. RESULT AND DISCUSSION

In Section 2, the construction of smooth surface by Catmull-Clark Subdivision scheme has been shown using a specific algorithm. Table 1 below shows the result of the surface by the number of iterations of a few sample models of polyhedron using the Catmull-Clark Subdivision scheme. In iteration zero phase, the shape is in its original state. Each of the shape has its own coordinates and faces and they are different with each other.

In iteration of phase one, the scheme is applied one time. Application of the scheme once again yields in iteration of phase two. Applying the scheme for the third time, resulting a smoother surface than the original. Hence, as the iteration increases, the polyhedron will produce smoother surface each time. As the scheme applied, cube and dodecahedron become almost sphere with different kinds of faces, which said to be smooth. Table 2 below

TABLE 1. Result of the surfaces by number of iteration

Number of iteration	Cube	Pyramid	Dodecahedron	Rhombic Hexecontahedron
0				
1				
2				
3				

summarizes the number of coordinates and faces corresponding to each model by number of iterations. As the number of iterations increases, the number of coordinates and faces increases and polyhedron become smoother (refer Table 1). In order to produce smooth surface, the scheme produce large number of new coordinates and faces as the iteration increases. Moreover, the number of all new faces are less two than the number of the new coordinates from first iteration to third iteration. Based on both tables shown, the minimum number of iteration for the polyhedron to produce smooth surface is three. At the third iteration of the model, there are no sharp corners or vertices can be seen anymore. Thus, the coordinates at the third iteration are the suitable coordinates to produce smooth surface by using Catmull-Clark subdivision scheme. Finally, subdivision of coordinates and faces are important criteria for smooth surface construction. Catmull-Clark subdivision scheme is able to produce smooth surface of polyhedron but it is very time-consuming to evaluate the coordinates until third iteration using Wolfram Mathematica 12.0 Student Edition on Asus X441N Series with Intel 2 Core N3350. As the coordinates keep increasing, Mathematica also takes a lot of time to evaluate 3842 coordinates for the Rhombic Hexecontahedron.

TABLE 2. Number of coordinates and faces for each polyhedron

Number of iteration	Number of	Cube	Pyramid	Dodecahedron	Rhombic Hexecontahedron
0	coordinates	8	5	20	62
	faces	6	5	12	60
1	coordinates	26	18	62	242
	faces	24	16	60	240
2	coordinates	98	66	242	962
	faces	96	64	240	960
3	coordinates	386	258	962	3842
	faces	284	256	960	3840

There are many other advantages of Catmull-Clark subdivision scheme apart from producing smooth surface of polyhedron. The scheme can also be used in constructing shapes with complex topology, such as shapes with holes in them. With appropriate modifications, Catmull-Clark subdivision surface algorithms can make it easier to construct complex shapes from a relatively simple control grid.

Besides, Catmull-Clark subdivision was developed in part to overcome the specific problem with Bézier patches and other patch-oriented solutions. This scheme allows animators to construct a single unified control grid for a complete complex figure. In addition, Catmull-Clark subdivision surface allows a higher level of control over shape than polygons and creases in conjunction with smooth and flowing surfaces to make the shape or model looks real mainly in character animation. Controllable sharpness creases can be accomplished using the generalization of the Catmull-Clark subdivision scheme suggested in [9].

However, Catmull-Clark subdivision scheme does face challenges because it does not interpolate its control net. The shape of the limit surface will be difficult to visualize from looking at the control net. The limitation is that as the net becomes denser, the surface will generally closer to the net and the detail of the new points will be closely-packed together. Thus the data processing also will be longer as it producing a large number of new points.

4. CONCLUSION

This research presented the construction of the classical subdivision surface which is Catmull-Clark subdivision scheme by using Wolfram Mathematica to produce smoother surface of a polyhedron. The construction starts from a standard polyhedron-cube and uses original coordinates as control points to produce new coordinates that have a smoother surface by iterative refinement scheme. The flexibility of this construction allows to construct any smoother surface from any original polyhedron with original coordinates and faces. For future research, the subdivision scheme can be improved in the computational iteration of the algorithm so that it might be able to produce smoother surface of any polyhedron without many iterations applied and less computational time. A full exploration of identifying the coordinates of smooth surface is another interesting future direction.

COMPETING INTERESTS

The authors declare that they have no competing interests.

ACKNOWLEDGEMENTS

The authors are grateful to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

AUTHOR'S CONTRIBUTIONS

All authors equally contributed to this work. All authors read and approved the final manuscript.

REFERENCES

[1] Chen, L. L., Zhang, Y., Lian, H., Atroshchenko, E., Ding, C., & Bordas, S. P. A. (2020). Seamless integration of computer-aided geometric modeling and acoustic simulation: Isogeometric boundary element methods based on Catmull-Clark subdivision surfaces. *Advances in Engineering Software*, **149**, 102879.

[2] Pan, Q., Rabczuk, T., Chen, C., Xu, G., & Pan, K. (2018). Isogeometric analysis of minimal surfaces on the basis of extended CatmullClark subdivision. *Computer Methods in Applied Mechanics and Engineering*, **337**, 128-149.

[3] Xie, J., Xu, J., Dong, Z., Xu, G., Deng, C., Mourrain, B., & Zhang, Y. J. (2020). Interpolatory Catmull-Clark volumetric subdivision over unstructured hexahedral meshes for modeling and simulation applications. *Computer Aided Geometric Design*, **80**, 101867.

[4] Catmull, E., & Clark, J. (1978). Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-aided design*, **10**(6), 350-355.

[5] Stam, J. (1998, July). Exact evaluation of Catmull-Clark subdivision surfaces at arbitrary parameter values. *In Proceedings of the 25th annual conference on Computer graphics and interactive techniques*, *1998*, 395-404.

[6] Yamaguchi, Y. (2001). A basic evaluation method of subdivision surfaces. *Journal for Geometry and Graphics*, **5**(2), 145-155.

[7] Bolz, J., & Schroder, P. (2002, February). Rapid evaluation of Catmull-Clark subdivision surfaces. *In Proceedings of the seventh international conference on 3D Web technology*, 11-17.

[8] Zeng, X. M., & Chen, X. J. (2006). Computational formula of depth for CatmullClark subdivision surfaces. Journal of computational and applied mathematics, **195**(1-2), 252-262.

[9] DeRose, T., Kass, M., & Truong, T. (1998, July). Subdivision surfaces in character animation. *In Proceedings of the 25th annual conference on Computer graphics and interactive techniques*, 85-94.

[10] Shen, J., Kosinka, J., Sabin, M. A., & Dodgson, N. A. (2014). Conversion of trimmed NURBS surfaces to CatmullClark subdivision surfaces. Computer Aided Geometric Design, **31**(7-8), 486-498.